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An introduction to relativistic spin hydrodynamics

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Abstract

Spin polarization and spin transport are common phenomena in many quantum systems. Relativistic spin hydrodynamics provides an effective low-energy framework to describe these processes in quantum many-body systems. The fundamental symmetry underlying relativistic spin hydrodynamics is angular momentum conservation, which naturally leads to inter-conversion between spin and orbital angular momenta. This inter-conversion is a key feature of relativistic spin hydrodynamics, which is closely related to entropy production and introduces ambiguity in the construction of constitutive relations. In this article, we present a pedagogical introduction of relativistic spin hydrodynamics. We demonstrate how to derive constitutive relations by applying local thermodynamic laws and explore several distinctive aspects of spin hydrodynamics. These include pseudo-gauge ambiguity, the behavior of the system in the presence of strong vorticity, and the challenges of modeling the freeze-out of spin in heavy-ion collisions. We also outline some future prospects for spin hydrodynamics.

Keywords Heavy-ion collision · Spin hydrodynamics · Spin polarization

1 Introduction

Spin is a fundamental property of particles arising from quantum mechanics, and it plays a central role in numerous phenomena within the quantum regime. As a form of angular momentum, spin naturally couples to rotation, allowing it to become polarized by rotational motion. Similarly, for a charged particle with nonzero spin, or a neutral particle with a non-trivial charge form factor, spin can couple to an external magnetic field as well. Additionally, for a particle in motion (i.e., with finite momentum), its spin may couple to acceleration, electric fields, or gradients of external potentials, such as chemical potential and temperature. In the case of massless particles, the spin state is specified by its helicity state, meaning that it is intrinsically slaved by the motion of the particle. As a result, spin can be manipulated by rotating fields, magnetic fields, electric fields, and several other external influences. Conversely, detecting the spin of a particle provides invaluable insights into the environment or underlying dynamics of the system.

In heavy-ion collision physics, the primary interest lies in the creation of deconfined quark-gluon matter, commonly referred to as the quark-gluon plasma (QGP) [1–5]. To uncover the properties of the QGP in heavy-ion collision experiments, it is essential to design specific hadronic observables that are sensitive to particular features of the QGP. Because charged particles are typically the easiest to detect, many observables rely on the charge of the hadrons. For instance, the total multiplicity of the detected charged hadrons reflects the initial energy of QGP. Meanwhile, the anisotropy in the momentum-space distribution of charged hadrons corresponds to the initial anisotropy in the spatial distribution of partons, leading to well-known harmonic flow parameters [6]. By measuring these hadronic observables, researchers have revealed several novel properties of

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hot and dense matter created during heavy-ion collisions. One significant finding is that the QGP must be extremely hot, with a typical temperature reaching 300-500 MeV at RHIC and LHC, indicating an extremely high energy density. Additionally, the QGP medium was found to interact strongly, with a very small shear viscosity to entropy density ratio η/s. This low ratio is required to explain the observed harmonic flow parameters [4, 5]. In fact, the η/s of the QGP is the lowest among all known fluids.

Since 2017, it has been established that the spin degree of freedom can be used to probe the properties of QGP [7]. This is achieved by measuring the spin polarization of spinful hadrons, such as hyperons and vector mesons [8–10]. Notably, it has been observed that the Λ and $\bar{\Lambda}$ hyperons can exhibit significant spin polarization at collision energies of tens of GeV [7, 11–14]. Similarly, the ϕ and J/ψ mesons exhibited considerable spin alignment [15, 16] [1]. These discoveries open new avenues for studying the QGP through the spin degree of freedom. For instance, we now understand that the global spin polarization (i.e., the total amount of spin polarization with respect to the reaction plane) of hyperons arises from angular momentum conservation through the formation of fluid vortices within the QGP: In non-central heavy-ion collisions, the system possesses substantial orbital angular momentum, which subsequently induces strong fluid vorticity in the QGP [17–19], thereby polarizing the spins of quarks via spin-rotation coupling [20–35]. However, to fully understand the spin polarization phenomenon, a dynamical theory of spin polarization and spin transport in a hot medium is essential, analogous to the necessity of a dynamical theory of the bulk medium for understanding harmonic flows. Naturally, such a dynamical theory of spin transport can be derived from either kinetic or hydrodynamic theory. In recent years, both spin kinetic theory and spin hydrodynamics have made significant advancements. In this article, we focus on spin hydrodynamics and refer readers to Refs. [36] for a review of spin kinetic theory and Refs. [37–44, 44–49] for a review of spin polarization phenomena in heavy-ion collisions. In addition, we will focus only on spin polarization in hot and dense medium rather than in systems created in, for example, electron-ion collisions [50].

Throughout this article, we use the natural units $c = \hbar = k_B = 1$ and the metric convention $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

[1] The spin alignment of a vector meson is quantified by the deviation of ρ_{00} from 1/3, where ρ_{00} is the 00-component of the vector meson's spin density matrix.

2 Relativistic hydrodynamics as an effective theory

Before discussing spin hydrodynamics, let us first briefly review the general structure of relativistic hydrodynamics from the perspective of effective field theory. The hydrodynamic theory describes the low-energy behavior of interacting many-body systems, where only conserved charge densities exhibit their dynamics. Because the conserved charge densities do not vanish, they redistribute themselves in space according to their equations of motion (EOMs). When expressed in a manner of spatial gradient expansion, these EOMs constitute hydrodynamic equations.

Let us consider the hydrodynamic theory of a system with space-time translation symmetry and global $U(1)$ symmetry. The corresponding conserved charge densities are the energy density $\varepsilon(x)$, momentum density $\pi^i(x), i = 1-3$, and the $U(1)$ charge density $n(x)$. We want to derive dynamical equations for these conserved charge densities. Sometimes, it is more convenient to work with potential variables conjugated to charge densities. These are the temperature $T(x)$ (or its inverse $\beta(x) = 1/T(x)$), fluid velocity $u^\mu(x)$ normalized as $u^\mu u^\nu \eta_{\mu\nu} = 1$, and chemical potential of $n(x)$, $\mu(x)$. These conserved charge densities (or equivalently their conjugates) are hydrodynamic variables in hydrodynamics. Our starting point is conservation laws:

$$\partial_\mu \Theta^{\mu\nu} = 0, \tag{1}$$

$$\partial_\mu J^\mu = 0, \tag{2}$$

where $\Theta^{\mu\nu}$ is the energy–momentum tensor, and J^μ is the $U(1)$ current. As an effective field theory, we express $\Theta^{\mu\nu}$ and J^μ in terms of the conserved charged density (or equivalently, their conjugates) and their various gradient orders. We assume spatial isotropy of the system, that is, there are no external forces breaking the $SO(3)$ symmetry. The building blocks are the fluid velocity u^μ and various quantities that can be classified into different representations of $SO(3)$ in the fluid rest frame. Up to the first-order gradients, these quantities are

$$\begin{aligned}
&\text{Scalar}: \quad \varepsilon,\ n,\ D\varepsilon, Dn, \theta \equiv \nabla \cdot u = \partial \cdot u,\\
&\text{Vector}: \quad Du^\mu,\ \nabla^\mu \varepsilon, \nabla^\mu n,\ \omega^{\mu\nu} \equiv -(1/2)(\nabla^\mu u^\nu - \nabla^\nu u^\mu),\\
&\text{Tensor}: \quad \sigma^{\mu\nu} \equiv (1/2)[\nabla^\mu u^\nu + \nabla^\nu u^\mu - (2/3)\Delta^{\mu\nu}\theta],
\end{aligned} \tag{3}$$

where $D \equiv u \cdot \partial$ is the co-moving time derivative, θ is the expansion rate of the fluid, $\nabla_\mu = \Delta_{\mu\nu}\partial^\nu$ is the spatial gradient operator, $\Delta_{\mu\nu} \equiv \eta_{\mu\nu} - u_\mu u_\nu$ is the spatial projector, $\sigma^{\mu\nu}$ is the shear tensor that is traceless, and $\omega_{\mu\nu}$ is the vorticity tensor. Note that the co-moving time derivatives will eventually be replaced by spatial gradients using the EOMs in the leading order. Note that the vorticity tensor transforms in the same way as a three-vector under proper three-rotations (i.e., a three-rotation R with $\det R = 1$) because it can be substituted by a three pseudo-vector $\omega^\mu \equiv -(1/2)\epsilon^{\mu\nu\rho\sigma}u_\nu\omega_{\rho\sigma}$. Consequently, we can write the most general structure decomposition up to $O(\partial)$ for $\Theta^{\mu\nu}$ and J^μ as follows [2]:

$$\begin{aligned}\Theta^{\mu\nu} = &(a_0 + b_0^\varepsilon D\varepsilon + b_0^n Dn + b_0^u\theta)u^\mu u^\nu \\ &+ c_0(u^\mu\nabla^\nu\varepsilon + u^\nu\nabla^\mu\varepsilon) + d_0(u^\mu\nabla^\nu n + u^\nu\nabla^\mu n) \\ &+ e_0(u^\mu Du^\nu + u^\nu Du^\mu) + f_0(u^\mu\omega^\nu + u^\nu\omega^\mu) \\ &+ (g_0 + h_0^\varepsilon D\varepsilon + h_0^n Dn + h_0^u\theta)\Delta^{\mu\nu} + i_0\sigma^{\mu\nu} \\ &+ j_0(u^\mu\nabla^\nu\varepsilon - u^\nu\nabla^\mu\varepsilon) + k_0(u^\mu\nabla^\nu n - u^\nu\nabla^\mu n) \\ &+ l_0(u^\mu Du^\nu - u^\nu Du^\mu) + m_0(u^\mu\omega^\nu - u^\nu\omega^\mu) \\ &+ n_0\epsilon^{\mu\nu\rho\sigma}u_\rho\nabla_\sigma\varepsilon + o_0\epsilon^{\mu\nu\rho\sigma}u_\rho\nabla_\sigma n + p_0\epsilon^{\mu\nu\rho\sigma}u_\rho Du_\sigma \\ &+ q_0\omega^{\mu\nu} + O(\partial^2),\end{aligned} \tag{4}$$

$$J^\mu = (A_0 + B_0^\varepsilon D\varepsilon + B_0^n Dn + B_0^u\theta)u^\mu + C_0\nabla^\mu\varepsilon + D_0\nabla^\mu n + E_0 Du^\mu + F_0\omega^\mu + O(\partial^2), \tag{5}$$

with all the coefficients (playing the roles of the Wilson coefficients in the effective field theory, as short-distance physics are encoded in these coefficients) functions of ε and n. They are constructed by first decomposing with respect to u^μ and then with respect to different representations of $SO(3)$. In these decompositions, the terms with f_0, m_0, n_0, o_0, p_0 in $\Theta^{\mu\nu}$ and F_0 in J^μ as coefficients transform differently from $\Theta^{\mu\nu}$ and J^μ under parity (P), respectively, meaning that they can appear only when the system contains parity violating content. Under time reversal transformation (T), all the terms of first-order gradients on the right-hand sides of $\Theta^{\mu\nu}$ and J^μ except for terms with coefficients $f_0, m_0, n_0, o_0, p_0, F_0$ transform differently from $\Theta^{\mu\nu}$ and J^μ, respectively. This means that these terms must be dissipative (i.e., these terms are responsible for entropy generation in the fluid), while terms with coefficients $f_0, m_0, n_0, o_0, p_0, F_0$ can appear without generating entropy, that is, they could arise in ideal hydrodynamics despite being at first order in gradients. Thus, the terms with coefficients $f_0, m_0, n_0, o_0, p_0, F_0$ are especially interesting. In fact, some of them have been intensively studied, and it was found that they contain very rich quantum phenomena (usually dubbed chiral anomalous transports) that are closely related to the chiral anomaly of the system if the underlying physics is governed by the quantum gauge theory. Recently, such chiral anomalous transport has become an active subject in condensed matter physics, astrophysics, and heavy-ion collision physics (see Refs. [41, 51–56] for recent reviews focusing on heavy-ion collision physics). Similarly, we can also examine the balance between the right-hand and left-hand sides of Eqs. (4)-(5) under charge conjugation (C) transformation. The terms with coefficients $b_0^n, d_0, h_0^n, k_0, o_0, A_0, B_0^\varepsilon, B_0^u, C_0, E_0, F_0$ must vanish if there is no environmental charge-conjugation violation (naturally, the presence of a nonzero charge density n violates the C symmetry and allows these terms to be present). The antisymmetric terms in $\Theta^{\mu\nu}$ are particularly interesting. To reveal their meaning, we consider the angular momentum conservation law:

$$\partial_\mu M^{\mu\nu\rho} = 0, \tag{6}$$

where $M^{\mu\nu\rho}$ is the angular momentum tensor

$$M^{\mu\nu\rho} = x^\nu\Theta^{\mu\rho} - x^\rho\Theta^{\mu\nu} + \Sigma^{\mu\nu\rho}, \tag{7}$$

and $\Sigma^{\mu\nu\rho}$ is the spin tensor. We can re-write Eq.(6) in the following form:

$$\partial_\mu\Sigma^{\mu\nu\rho} = \Theta^{\rho\nu} - \Theta^{\nu\rho}. \tag{8}$$

Thus, the antisymmetric part of $\Theta^{\mu\nu}$ provides a source for spin generation (one may more clearly see this by integrating Eq.(8) over space). This will be the focus of this study, and we return to it in the next section. In the remainder of this section, for the purpose of demonstrating the construction of the hydrodynamic theory, we simply assume that the system does not possess a spin tensor, so that $\Theta^{\mu\nu}$ is symmetric, $\Theta^{\mu\nu} = \Theta^{\nu\mu}$, and assume that there is no environmental parity violation, so that terms with coefficients $f_0, m_0, n_0, o_0, p_0, F_0$ must vanish. Thus, the most general decomposition of $\Theta^{\mu\nu}$ and J^μ up to the first order in gradients into different components with respect to u^μ, and subsequently, for the components orthogonal to u^μ, with respect to different irreducible tensor structures under $SO(3)$ are as follows:

$$\begin{aligned}\Theta^{\mu\nu} = &(a_0 + b_0^\varepsilon D\varepsilon + b_0^n Dn + b_0^u\theta)u^\mu u^\nu \\ &+ c_0(u^\mu\nabla^\nu\varepsilon + u^\nu\nabla^\mu\varepsilon) + d_0(u^\mu\nabla^\nu n + u^\nu\nabla^\mu n) + e_0(u^\mu Du^\nu + u^\nu Du^\mu) \\ &+ (g_0 + h_0^\varepsilon D\varepsilon + h_0^n Dn + h_0^u\theta)\Delta^{\mu\nu} \\ &+ i_0\sigma^{\mu\nu} \\ &+ O(\partial^2),\end{aligned} \tag{9}$$

$$\begin{aligned}J^\mu = &(A_0 + B_0^\varepsilon D\varepsilon + B_0^n Dn + B_0^u\theta)u^\mu + C_0\nabla^\mu\varepsilon \\ &+ D_0\nabla^\mu n + E_0 Du^\mu + O(\partial^2).\end{aligned} \tag{10}$$

Up to this point, expressions (9)–(10) are merely parameterizations of $\Theta^{\mu\nu}$ and J^μ, and such a parameterization is ambiguous at $O(\partial)$ order (and higher orders in gradients). To see this, we consider to re-express $\Theta^{\mu\nu}$ and J^μ in terms of a redefinition of the hydrodynamic variables ε', n', u'^μ which differ from ε, n, u^μ by $O(\partial)$-order shifts:

$$\varepsilon' = \varepsilon + \delta\varepsilon, \quad n' = n + \delta n, \quad u'^\mu = u^\mu + \delta u^\mu, \tag{11}$$

[2] One can start without including the co-moving time-derivative terms as those terms are eventually replaced by the spatial gradients up on using leading-order hydrodynamic EOMs. But we keep them to make the discussions more transparent.

where $\delta\varepsilon, \delta n, \delta u^\mu$ are order-$O(\partial)$ quantities and $u_\mu \delta u^\mu = O(\partial^2)$ so that $u'^2 = 1$ is maintained at $O(\partial)$. This can be seen by noting that $u'^2 = u^2 + 2u_\mu \delta u^\mu + \delta u^2$ which leads to $2u_\mu \delta u^\mu + \delta u^2 = O(\partial^2)$ and therefore $u_\mu \delta u^\mu$ must be $O(\partial^2)$. In terms of the primed variables, we obtain

$$\begin{aligned}\Theta^{\mu\nu} =&\left[a'_0 - \left(\frac{\partial a_0}{\partial \varepsilon}\delta\varepsilon + \frac{\partial a_0}{\partial n}\delta n\right) + b_0^\varepsilon D\varepsilon + b_0^n Dn + b_0^u \theta\right] u'^\mu u'^\nu \\ &+ c_0(u^\mu \nabla^\nu \varepsilon + u^\nu \nabla^\mu \varepsilon) + d_0(u^\mu \nabla^\nu n + u^\nu \nabla^\mu n) + \\ &e_0(u^\mu D u^\nu + u^\nu D u^\mu) + (g_0 - a_0)(\delta u^\mu u^\nu + u^\mu \delta u^\nu) \\ &+ \left[g'_0 - \left(\frac{\partial g_0}{\partial \varepsilon}\delta\varepsilon + \frac{\partial g_0}{\partial n}\delta n\right) + h_0^\varepsilon D\varepsilon + h_0^n Dn + h_0^u \theta\right]\Delta'^{\mu\nu} \\ &+ i_0 \sigma^{\mu\nu} + O(\partial^2),\end{aligned} \tag{12}$$

$$\begin{aligned}J^\mu =&\left[A'_0 - \left(\frac{\partial A_0}{\partial \varepsilon}\delta\varepsilon + \frac{\partial A_0}{\partial n}\delta n\right) + B_0^\varepsilon D\varepsilon + B_0^n Dn + B_0^u \theta\right] u'^\mu \\ &- A_0 \delta u^\mu + C_0 \nabla^\mu \varepsilon + D_0 \nabla^\mu n + E_0 D u^\mu \\ &+ O(\partial^2),\end{aligned} \tag{13}$$

where $a'_0 = a_0(\varepsilon', n')$ and similarly for g'_0, A'_0 and all second-order terms are omitted. By observing the expressions in the three square brackets, one can see that by suitably choosing $\delta\varepsilon$ and δn, one can eliminate the first-order terms in two of the three square brackets. For example, one can solve out $\delta\varepsilon$ and δn by requiring the first-order terms in square brackets in $\Theta^{\mu\nu}$ to vanish. However, it is more convenient to eliminate the first-order terms in the coefficients of $u'^\mu u'^\nu$ in $\Theta^{\mu\nu}$ and u'^μ in J^μ. Similarly, by suitably choosing δu^μ, one can eliminate either the second line in $\Theta^{\mu\nu}$ (this choice is called the Landau–Lifshitz frame for u^μ) or the second line in J^μ (this choice is called the Eckart frame for u^μ). Therefore, we can always choose the following simpler forms for $\Theta^{\mu\nu}$ and J^μ (Landau–Lifshitz frame),

$$\Theta^{\mu\nu} = a_0 u^\mu u^\nu + (g_0 + h_0^\varepsilon D\varepsilon + h_0^n Dn + h_0^u \theta)\Delta^{\mu\nu} + i_0 \sigma^{\mu\nu} + O(\partial^2), \tag{14}$$

$$J^\mu = A_0 u^\mu + C_0 \nabla^\mu \varepsilon + D_0 \nabla^\mu n + E_0 D u^\mu + O(\partial^2). \tag{15}$$

Contracting with u^μ, we can identify that $a_0 = u_\mu u_\nu \Theta^{\mu\nu}$ which is the local energy density ε and $A_0 = u \cdot J$ which is the local $U(1)$ charge density n. [Sometimes, this is also considered as the matching condition because this means that $u_\mu u_\nu \Theta^{\mu\nu} = u_\mu u_\nu \Theta^{\mu\nu}_{(0)}$ and $u_\mu J^\mu = u_\mu J^\mu_{(0)}$ with $\Theta^{\mu\nu}_{(0)}$ and $J^\mu_{(0)}$ is the zeroth order energy–momentum tensor and charge current.]

Let us first consider the zeroth-order terms, which, as we have already discussed, correspond to ideal hydrodynamics:

$$\Theta^{\mu\nu}_{(0)} = \varepsilon u^\mu u^\nu + g_0 \Delta^{\mu\nu}, \tag{16}$$

$$J^\mu_{(0)} = n u^\mu. \tag{17}$$

In the rest frame of the fluid, u^μ=(1, $\mathbf{0}$), it becomes $\Theta^{\mu\nu}_{(0)} = \mathrm{diag}(\varepsilon, -g_0, -g_0, -g_0)$ which identifies $-g_0$ as the thermodynamic pressure P. In the zeroth order, the conservation laws are

$$(\varepsilon + P) D u^\mu - \nabla^\mu P = 0, \tag{18}$$

$$D\varepsilon + (\varepsilon + P)\theta = 0, \tag{19}$$

$$Dn + n\theta = 0. \tag{20}$$

To close these equations, we need to know the thermodynamic relation among P, ε, n, that is, the equation of state, $P = P(\varepsilon, n)$.

Let us then consider the first-order terms that correspond to dissipative hydrodynamics. From Eqs. (18)-(20), we notice that we could replace $D\varepsilon$ and Dn in the first order terms by $-(\varepsilon + P)\theta$ and $-n\theta$ and Du^μ by $\nabla^\mu P/(\varepsilon + P)$. This allows us to re-write the energy–momentum tensor and charge current as

$$\Theta^{\mu\nu} = \varepsilon u^\mu u^\nu - (P + h_0 \theta)\Delta^{\mu\nu} + i_0 \sigma^{\mu\nu} + O(\partial^2), \tag{21}$$

$$J^\mu = n u^\mu + C'_0 \nabla^\mu \varepsilon + D'_0 \nabla^\mu n + O(\partial^2), \tag{22}$$

with $h_0 = (\varepsilon + P)h_0^\varepsilon + n h_0^n - h_0^u$, $C'_0 = C_0 + E_0(\partial P/\partial \varepsilon)_n/(\varepsilon + P)$, and $D'_0 = D_0 + E_0(\partial P/\partial n)_\varepsilon/(\varepsilon + P)$. Further constraints can be imposed, based on the laws of local thermodynamics. For a fluid at rest, we have the first law of thermodynamics as

$$T\mathrm{d}s + \mu \mathrm{d}n = \mathrm{d}\varepsilon, \tag{23}$$

$$Ts + \mu n = \varepsilon + P, \tag{24}$$

where s denotes the entropy density. To proceed, we propose the covariant generalization of the second one (Gibbs–Duhem relation):

$$s^\mu = P\beta^\mu + \Theta^{\mu\nu}\beta_\nu - \alpha J^\mu, \tag{25}$$

where $\beta^\mu = \beta u^\mu$ ($\beta = 1/T$), $\alpha = \mu/T$, and s^μ is the entropy current, such that $u \cdot s = s$. The divergence of s^μ (multiplied by T) can be calculated directly as:

$$T\partial_\mu s^\mu = \Theta^{\mu\nu}_{(1)} \nabla_\mu u_\nu - T J^\mu_{(1)} \nabla_\mu \alpha. \tag{26}$$

The second law of local thermodynamics requires that $T\partial_\mu s^\mu \geq 0$ for any configuration of the velocity field u^μ, temperature T, and chemical potential μ, which imposes the following constraints:

$$h_0 = -\zeta \leq 0, \quad i_0 = 2\eta \geq 0, \quad J^\mu_{(1)} = \sigma \nabla^\mu \alpha, \tag{27}$$

where ζ and η are the bulk and shear viscosities, respectively, and σ is charge conductivity. This also shows that the coefficients C'_0 and D'_0 are fixed in such a way that $C'_0\nabla^\mu\varepsilon + D'_0\nabla^\mu n = \sigma\nabla^\mu\alpha$. The EOMs of the first-order dissipative hydrodynamics are then read

$$(\varepsilon + P - \zeta\theta)Du^\mu - \nabla^\mu(P - \zeta\theta) + 2\eta\Delta^\mu_\nu\partial_\rho\sigma^{\nu\rho} = 0\,, \tag{28}$$

$$D\varepsilon + (\varepsilon + P - \zeta\theta)\theta - 2\eta\sigma_{\mu\nu}\sigma^{\mu\nu} = 0, \tag{29}$$

$$Dn + n\theta + \sigma\nabla^2\alpha = 0. \tag{30}$$

The first equation is the relativistic Navier–Stokes equation. The above procedure can continue to a higher order in gradients and provide higher-order hydrodynamics. However, we did not discuss these more complicated situations. Readers can find discussions in Refs. [57–61].

3 Construction of relativistic spin hydrodynamics

With the above preparation, we now discuss the construction of relativistic spin hydrodynamics, in which the conservation of angular momentum is explicitly encoded within a (quasi)-hydrodynamic framework. The fundamental conservation laws are the energy–momentum conservation (1) and angular momentum conservation (8). Before delving into the detailed construction, we note that if we assign spin density $S^{\mu\nu} = u_\rho\Sigma^{\rho\mu\nu}$ as a dynamic variable in our framework, Eq. (8) indicates that it is generally not conserved. This reflects the fact that the spin angular momentum can be transformed into orbital angular momentum, thus disqualifying it as a true hydrodynamic mode. Consequently, spin hydrodynamics is not a strict hydrodynamic theory for the gapless modes. Instead, it should be categorized as quasi-hydrodynamics, where the low-energy dynamic variables comprise true hydrodynamic modes and some gapped modes (quasi-hydrodynamic modes) whose gap in the low-momentum region is parametrically small compared with other microscopic modes (the hard modes of the system) [62]. This results in spectrum separation; for physics at energy scales comparable to these modes, we can only consider the quasi-hydrodynamic modes alongside the true hydrodynamic modes. Generalized hydrodynamics [63] and Hydro+ [64] near the QCD critical point fall into this category. The spin hydrodynamics that we will discuss also belongs to this type of theory. This framework requires that spin excitations, despite being gapped, remain low-energy excitations compared with other microscopic modes [62]. For instance, if the system contains massive fermions, the spins of these fermions are difficult to relax because the spin-orbit coupling is inversely suppressed by the mass of the fermions compared to the typical energy transfer [65–67]. Thus, these spins are quasi-conserved, and we can formulate a quasi-hydrodynamic theory for it, which is called spin hydrodynamics.

We consider a charge-neutral system such as the quark gluon plasma or the usual electric plasma, in which some of the constituent particles are spinful particles. The symmetry considered is space-time translation symmetry and Lorentz symmetry. This leads to the energy–momentum conservation and angular momentum conservation, as given by Eq.(1) and Eq.(8). Now, the spin tensor $\Sigma^{\mu\rho\sigma}$ plays the role of the charge current J^μ and we can write it as $\Sigma^{\mu\rho\sigma} = S^{\rho\sigma}u^\mu + \text{higher order terms}$, with the spin density $S^{\rho\sigma}$ playing a similar role to the charge density n in Eq.(22). To proceed, we need to choose a suitable power-counting scheme for all (quasi-)hydrodynamic variables. If we consider the QGP in heavy ion collisions, from the measurements of global spin polarization of hyperons, we know that the spin density in the QGP should be small because the hyperon spin polarization is only a few percent. Thus, it is reasonable to assume that the spin density $S^{\rho\sigma}$ is parametrically smaller than the true hydrodynamic modes described by variables ε and u^μ. Thus, we take the following power-counting scheme:

$$\varepsilon, P, T, u^\mu \sim O(1), \tag{31}$$

$$S^{\rho\sigma} \sim O(\partial). \tag{32}$$

Analogous to the fact that the chemical potential μ is conjugate to the charge density n, we can introduce the spin potential $\mu^{\rho\sigma}$ to be conjugate thermodynamically to the spin density $S^{\rho\sigma}$ and *propose* the first law for local thermodynamics as (analogous to Eq.(23)):

$$Tds + \frac{1}{2}\mu_{\mu\nu}dS^{\mu\nu} = d\varepsilon, \tag{33}$$

$$Ts + \frac{1}{2}\mu_{\mu\nu}S^{\mu\nu} = \varepsilon + P. \tag{34}$$

Following the discussions on the fluid local frame, we realize that the same discussions are still valid for the symmetric part of the energy–momentum tensor; thus, we still choose the definition of u^μ such that it is the eigenvector of the symmetric part of the energy–momentum tensor, $\Theta^{\mu\nu}_s$ (we still call it the Landau–Lifshitz frame) [3]:

[3] Since $S^{\rho\sigma}$ is counted as $O(\partial)$ quantities, the term $S^{\rho\sigma}u^\mu$ is unchanged at $O(\partial)$ under a re-definition of $u^\mu \to u^\mu + \delta u^\mu$ with $\delta u^\mu \sim O(\partial)$. Therefore, Eq.(35) is automatically satisfied at $O(\partial)$ up on using the zeroth-order EOM for u^μ [68]. But when there appear other conserved charges, such as a global $U(1)$ charge, Eq.(35) is a proposal to fix the

$$\Theta_s^{\mu\nu} u_\nu = \varepsilon u^\mu. \tag{35}$$

Because the EOM for spin density involves only the antisymmetric part of the energy–momentum tensor $\Theta_a^{\mu\nu}$, the symmetric part $\Theta_s^{\mu\nu}$, still takes the same tensor decomposition up to the first order in gradients, as in Eq.(21):

$$\Theta_s^{\mu\nu} = \varepsilon u^\mu u^\nu - (P - \zeta\theta)\Delta^{\mu\nu} + 2\eta\sigma^{\mu\nu} + O(\partial^2). \tag{36}$$

To determine the form of $\Theta_a^{\mu\nu}$, we used the second law of local thermodynamics. The covariant entropy current is (an analog of Eq.(25))

$$s^\mu = P\beta^\mu + \Theta^{\mu\nu}\beta_\nu - \frac{1}{2}\alpha_{\rho\sigma}\Sigma^{\mu\rho\sigma}, \tag{37}$$

with $\alpha_{\rho\sigma} = \mu_{\rho\sigma}/T$. The production rate of entropy then reads

$$T\partial_\mu s^\mu = \Theta_{s(1)}^{\mu\nu}\partial_{(\mu}u_{\nu)} + \Theta_a^{\mu\nu}\left(\mu_{\mu\nu} + T\partial_{[\mu}\beta_{\nu]}\right) + O(\partial^3). \tag{38}$$

The semi-positiveness of the first term on the right-hand side is guaranteed when both the bulk and shear viscosities are semi-positive. The requirement of the semi-positiveness of the second term gives the constitutive relation for $\Theta_a^{\mu\nu}$ at $O(\partial)$ order [68]

$$\Theta_a^{\mu\nu} = q^\mu u^\nu - q^\nu u^\mu + \phi^{\mu\nu}, \tag{39}$$

$$q^\mu = \lambda\left[\beta\nabla^\mu T + Du^\mu - 2\mu^{\mu\nu}u_\nu\right], \tag{40}$$

$$\phi^{\mu\nu} = \eta_s\Delta^{\mu\rho}\Delta^{\nu\sigma}\left(\mu_{\rho\sigma} - T\varpi_{\rho\sigma}\right). \tag{41}$$

The quantity

$$\varpi_{\mu\nu} = (1/2)\left(\partial_\nu\beta_\mu - \partial_\mu\beta_\nu\right) \tag{42}$$

is the thermal vorticity tensor. The quantities λ and η_s must be semi-positive to guarantee the semi-positivity of the entropy production. These are called boost heat conductivity and rotational viscosity, respectively [68]. Using these constitutive relations, we obtain the spin hydrodynamic equations up to $O(\partial^2)$ order:

$$\begin{aligned}&(\varepsilon + P - \zeta\theta)Du^\mu - \nabla^\mu(P - \zeta\theta) + 2\eta\Delta^\mu_\nu\partial_\rho\sigma^{\nu\rho}\\ &+ q\cdot\partial u^\mu - \Delta^\mu_\nu Dq^\nu - q^\mu\theta + \Delta^\mu_\rho\partial_\nu\phi^{\nu\rho} = 0\,,\end{aligned} \tag{43}$$

$$\begin{aligned}&D\varepsilon + (\varepsilon + P - \zeta\theta)\theta - 2\eta\sigma_{\mu\nu}\sigma^{\mu\nu}\\ &+ \partial\cdot q + q_\mu Du^\mu + \phi^{\mu\nu}\omega_{\mu\nu} = 0,\end{aligned} \tag{44}$$

$$DS^{\rho\sigma} + S^{\rho\sigma}\theta + 2\Theta_a^{\rho\sigma} = 0. \tag{45}$$

Footnote 3 (continued)
local rest frame of the fluid.

In this section, we present a detailed derivation of the constitutive relations for relativistic spin hydrodynamics up to first order. For related discussions that follow a similar approach, see Refs. [62, 69–78]. Other methodologies for deriving and analyzing the constitutive relations of spin hydrodynamics have also been discussed in the literature, including utilizing the hydrostatic partition function with constraints from the entropy current and Onsager relations [79, 80], using local equilibrium and non-equilibrium statistical operators [73, 77, 81–83], and employing kinetic theories [76, 84–95]. Relativistic spin hydrodynamics have become a vibrant area of research, attracting intense discussion in recent years. In the following section, we will explore some of these developments; further insights can be found in Refs. [96–110].

4 Discussions

We developed spin hydrodynamics based on local thermodynamic laws. Spin hydrodynamics exhibit several novel features that differ significantly from those found in conventional relativistic hydrodynamics for other types of conservation laws (e.g., the energy–momentum conservation and baryon number conservation). In this section, we explore and discuss certain intriguing characteristics.

4.1 Pseudo-gauge ambiguity

The definition of conserved current is not unique. One example is the magnetization current and dipole charge density. Let $J^\mu = (\rho, \boldsymbol{J})$ represent the conserved conductive electric current. For a polarizable and magnetizable material, the total charge density and electric current are given by $\tilde{\rho} = \rho + \boldsymbol{\nabla}\cdot\boldsymbol{P}$ and $\tilde{\boldsymbol{J}} = \boldsymbol{J} + \boldsymbol{\nabla}\times\boldsymbol{M}$, respectively, where $\boldsymbol{P}$ is the electric dipole density, and $\boldsymbol{M}$ is the magnetization density. In the covariant form, we have:

$$\tilde{J}^\mu = J^\mu + \partial_\nu\mathcal{M}^{\mu\nu} \quad \text{with} \quad \mathcal{M}^{\mu\nu} = -\mathcal{M}^{\nu\mu}. \tag{46}$$

Obviously, the total current $\tilde{J}^\mu$ is conserved if the conduction current J^μ is conserved, and the total electric charge remains unchanged provided the surface dipole density vanishes. The transformation of a conserved current that preserves both the original conservation law and total conserved charge is called a *pseudo-gauge transformation*. The example above demonstrates that the total current and conduction current differ by a pseudo-gauge transformation (with the magnetization $\mathcal{M}^{\mu\nu}$ serving as the pseudo-gauge field). This example also highlights that a pseudo-gauge transformation is not a true gauge transformation because it alters the physical content of the transformed current. Further insight into the pseudo-gauge transformation can be obtained by examining Maxwell's equations:

$$\partial_\mu F^{\mu\nu} = \tilde{J}^\nu. \tag{47}$$

One could subtract $-\partial_\rho \mathcal{M}^{\nu\rho}$ from both sides and find

$$\partial_\mu H^{\mu\nu} = J^\nu, \tag{48}$$

where the new field-strength tensor is defined as $H^{\mu\nu} \equiv F^{\mu\nu} + \mathcal{M}^{\mu\nu}$. This demonstrates that without imposing additional constraints, the two sets of fields, $(F^{\mu\nu}, \tilde{J}^\mu)$ and $(H^{\mu\nu}, J^\mu)$, describe the same physical laws, and one can freely choose which set to use. (If further constraints are imposed, such as the Bianchi equation $\partial_{[\rho} F_{\mu\nu]} = 0$, which is not preserved under a general pseudo-gauge transformation, then only certain pseudo-gauges that respect the Bianchi equation are permitted.)

Similarly, let us consider angular momentum conservation (note the analogy with Eq.(48), where $\Sigma^{\mu\nu\rho}$ and $\Theta^{\rho\nu} - \Theta^{\nu\rho}$ are analogous to $H^{\mu\nu}$ and J^μ in Eq.(48)):

$$\partial_\mu \Sigma^{\mu\nu\rho} = \Theta^{\rho\nu} - \Theta^{\nu\rho}, \tag{49}$$

which is preserved under the transformation

$$\Sigma^{\mu\rho\sigma} \to \tilde{\Sigma}^{\mu\rho\sigma} \equiv \Sigma^{\mu\rho\sigma} - \Phi^{\mu\rho\sigma}, \tag{50}$$

$$\Theta^{\mu\nu} \to \tilde{\Theta}^{\mu\nu} \equiv \Theta^{\mu\nu} + \frac{1}{2}\partial_\lambda \Phi^{\lambda\mu\nu}, \tag{51}$$

with $\Phi^{\lambda\mu\nu} = -\Phi^{\lambda\nu\mu}$ denotes an arbitrary local field. However, this transformation violates the conservation law of the energy–momentum tensor. It can be modified as follows:

$$\Sigma^{\mu\rho\sigma} \to \tilde{\Sigma}^{\mu\rho\sigma} \equiv \Sigma^{\mu\rho\sigma} - \Phi^{\mu\rho\sigma}, \tag{52}$$

$$\Theta^{\mu\nu} \to \tilde{\Theta}^{\mu\nu} \equiv \Theta^{\mu\nu} + \frac{1}{2}\partial_\lambda \left(\Phi^{\lambda\mu\nu} - \Phi^{\mu\lambda\nu} - \Phi^{\nu\lambda\mu}\right), \tag{53}$$

which preserves Eq.(49) and Eq.(1). Given a spacelike hypersurface Ξ, the total energy–momentum and total angular momentum across Ξ are

$$P^\nu = \int d\Xi_\mu \Theta^{\mu\nu}, \tag{54}$$

$$\begin{aligned} M^{\rho\sigma} &= \int d\Xi_\mu M^{\mu\rho\sigma} \\ &= \int d\Xi_\mu (x^\rho \Theta^{\mu\sigma} - x^\sigma \Theta^{\mu\rho} + \Sigma^{\mu\rho\sigma}). \end{aligned} \tag{55}$$

One can check that P^μ and $M^{\rho\sigma}$ are invariant under pseudo-gauge transformation (52) and (53) if the pseudo-gauge field $\Phi^{\mu\rho\sigma}$ vanishes at the boundary of Ξ [4].

[4] This can be checked by noting that for $A^{\lambda\mu\nu} = -A^{\mu\lambda\nu}$ we have $\int d\Xi_\mu \partial_\lambda A^{\lambda\mu\nu} = \int d\Xi_\mu \partial_\lambda^\perp A^{\lambda\mu\nu} + \int d\Xi n_\mu n_\lambda n \cdot \partial A^{\lambda\mu\nu} = \int d\Xi_\mu \partial_\lambda^\perp A^{\lambda\mu\nu}$ with n^μ the norm of Ξ and $\partial_\lambda^\perp = \partial_\lambda - n_\lambda n \cdot \partial$. Then, one can use the Gauss theorem to transform it to an integral over the boundary of Ξ.

One consequence of the pseudo-gauge transformation is the freedom to choose the symmetry properties of the spin tensor. To illustrate this, we consider an example in which we aim to transform the general spin tensor $\Sigma^{\mu\rho\sigma} = -\Sigma^{\mu\sigma\rho}$ into a completely antisymmetric form. We can choose $\Phi^{\mu\rho\sigma} = \Sigma^{(\mu\rho)\sigma} - \frac{1}{2}\Sigma^{\sigma\mu\rho}$. After applying the pseudo-gauge transformation, we obtain

$$\Sigma^{\mu\rho\sigma} \to \tilde{\Sigma}^{\mu\rho\sigma} = \frac{1}{2}(\Sigma^{\mu\rho\sigma} - \Sigma^{\rho\mu\sigma} + \Sigma^{\sigma\mu\rho}), \tag{56}$$

$$\Theta^{\mu\nu} \to \tilde{\Theta}^{\mu\nu} = \Theta^{\mu\nu} + \frac{1}{4}\partial_\lambda \left(3\Sigma^{\nu\mu\lambda} + \Sigma^{\mu\nu\lambda} - \Sigma^{\lambda\nu\mu}\right). \tag{57}$$

Note that the obtained $\tilde{\Sigma}^{\mu\rho\sigma}$ is totally antisymmetric; therefore, it can be parameterized as

$$\tilde{\Sigma}^{\mu\rho\sigma} = -\epsilon^{\mu\rho\sigma\nu}\tilde{S}_\nu, \tag{58}$$

where $\tilde{S}^\mu$ denotes the corresponding spin (pseudo)vector. Thus, the spin density tensor is thus $\tilde{S}^{\mu\nu} = -\epsilon^{\mu\nu\rho\sigma}u_\rho \tilde{S}_\sigma$. The main difference between this spin density tensor and that used in Sect. 3 is that $\tilde{S}^{\mu\nu}$ contains three degrees of freedom corresponding to the three spatial spin vectors, whereas $S^{\mu\nu}$ has six degrees of freedom, with three for spatial spin and three for boost. Thus, in some cases, it is more convenient to use $\tilde{\Sigma}^{\mu\rho\sigma}$ to construct the spin hydrodynamics. By following a procedure similar to that adopted in Sect. 3, we can derive constitutive relations in this context. In doing so, we decompose $\tilde{S}^\mu$ into $\tilde{S}^\mu = \sigma^\mu + n_5 u^\mu$, where σ^μ represents the spatial spin with the condition $\sigma \cdot u = 0$, and n_5 is a pseudoscalar field (hence subscript 5). We also decompose $\tilde{\Theta}^{\mu\nu}$ into:

$$\tilde{\Theta}^{\mu\nu} = \varepsilon u^\mu u^\nu - P\Delta^{\mu\nu} + \tilde{\Theta}^{\mu\nu}_{s(1)} + \tilde{q}^\mu u^\nu - \tilde{q}^\nu u^\mu + \tilde{\phi}^{\mu\nu}, \tag{59}$$

where, as in Sect. 3, we assumed the Landau–Lifshitz frame

$$\tilde{\Theta}^{\mu\nu}_s u_\nu = \varepsilon u^\mu, \tag{60}$$

such that $\tilde{\Theta}^{\mu\nu}_{s(1)}$ is purely transverse to u^μ. It is important to note that although we use the same symbols ε, P, and u^μ as in Sect. 3, their actual values may differ because the energy–momentum tensors and spin tensors in these two cases are different (but connected by the pseudo-gauge transformations (56) and (57)). We adopt a power counting scheme similar to the one we chose in Sect. 3,

$$\varepsilon, P, T, u^\mu \sim O(1), \tag{61}$$

$$\tilde{S}^\mu, \tilde{q}^\mu, \tilde{\phi}^{\mu\nu} \sim O(\partial). \tag{62}$$

Using the same form for the entropy current and the first law for local thermodynamics presented in Sect. 3, one can then find the divergence of the entropy current to be

$$T\partial_\mu s^\mu = \tilde{\Theta}^{\mu\nu}_{s(1)}\partial_{(\mu}u_{\nu)} + \tilde{\Theta}^{\mu\nu}_{a}\big(\tilde{\mu}_{\mu\nu} + T\partial_{[\mu}\beta_{\nu]}\big) + O(\partial^3). \tag{63}$$

We note that in deriving this result, we have utilized the fact that contracting the equation of motion (49) with u^ρ reveals that $\tilde{q}^\mu$ is not an independent current, but is determined by $\tilde{S}^\mu$ through the following relation:

$$\tilde{q}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu\nabla_\rho\tilde{S}_\sigma. \tag{64}$$

This is because when the spin tensor is completely antisymmetric, the components responsible for the boost are gauged away, meaning that the corresponding torque for the boost in the antisymmetric part of the energy–momentum tensor cannot be an independent current either. Owing to this relationship, we can show that $n_5 = \tilde{S}\cdot u$ is actually an $O(\partial^3)$ quantity (and thus does not appear on the right-hand side of Eq.(63)). In fact, through direct calculation, one can find that the higher order terms that are neglected in Eq.(63) contains only one term: $\propto n_5$,

$$\frac{1}{2}n_5\epsilon^{\mu\nu\rho\sigma}u_\sigma\big[\partial_\mu(\beta\tilde{\mu}_{\nu\rho}) + \nabla_\mu u_\rho(\partial_\nu\beta + D\beta_\nu)\big], \tag{65}$$

which infers that $n_5 \propto \epsilon^{\mu\nu\rho\sigma}u_\sigma\big[\partial_\mu(\beta\tilde{\mu}_{\nu\rho}) + \nabla_\mu u_\rho(\partial_\nu\beta + D\beta_\nu)\big] \sim O(\partial^3)$ and thus can be neglected [62]. Therefore, from Eq.(63), we derive the constitutive relations for spin hydrodynamics with a completely antisymmetric spin tensor as follows [62]:

$$\tilde{\Theta}^{\mu\nu}_{s(1)} = \zeta\theta\Delta^{\mu\nu} + 2\eta\sigma^{\mu\nu}, \tag{66}$$

$$\tilde{\Theta}^{\mu\nu}_{a(1)} = \tilde{\phi}^{\mu\nu}_{(1)} = \eta_s\Delta^{\mu\rho}\Delta^{\nu\sigma}\big(\mu_{\rho\sigma} - T\varpi_{\rho\sigma}\big). \tag{67}$$

Although these relations take the same form as those obtained in Sect. 3, it is important to note that they apply specifically to the pseudo-gauge of a completely antisymmetric spin tensor. These relations are particularly convenient for describing the evolution of spatial spin degrees of freedom.

Thus, choosing different forms for the spin tensor (loosely referred to as different pseudo-gauges) leads to different forms for the constitutive relations. In an extreme case, one might even select $\Phi^{\mu\rho\sigma} = \Sigma^{\mu\rho\sigma}$, which completely eliminates the spin tensor and renders the energy–momentum tensor totally symmetric (this choice is commonly referred to as the Belinfante gauge [111–113]). While this may seem to eliminate all information about spin in hydrodynamics, the energy density, viscous tensors, and heat current remain influenced by spin, meaning that the dynamics of spin are still embedded within these quantities. For discussions on the transformation from canonical to Belinfante gauges, see Refs. [69, 71, 75, 99, 114, 115]. In addition, other pseudo-gauges have been employed and discussed in the context of spin hydrodynamics [85, 86, 88, 116–119].

4.2 Spin hydrodynamics for strong vorticity

The power counting scheme employed in the previous discussions is motivated by the observation that at global equilibrium, the spin potential $\mu_{\mu\nu}$ is determined by the thermal vorticity $\varpi_{\mu\nu} = (\partial_\nu\beta_\mu - \partial_\mu\beta_\nu)/2$, which is naturally assumed to be an $O(\partial)$ quantity. However, this assumption may not hold true because the global equilibrium allows for arbitrarily large rotations (vorticity). When the vorticity is large, the assignment $\varpi_{\mu\nu} \sim O(\partial)$ becomes inadequate; instead, it is more appropriate to consider that $\varpi_{\mu\nu} \sim O(1)$. We explore this situation in this subsection, following closely the discussions in Ref. [72]. Before going into the details, it is useful to compare spin hydrodynamics with magnetohydrodynamics (MHD), in which the magnetic field is treated as an $O(1)$ quantity (See Ref. [120] for a review of relativistic MHD).

The MHD describes the coupled evolution of fluid energy–momentum (or temperature and velocity) and the electromagnetic field in the low-energy and long-wavelength regimes. The fundamental equations consist of the conservation laws for the energy–momentum tensor and Maxwell's equations. Owing to the screening effect, the electric fields within the fluid are gapped and parametrically small compared to the magnetic field. This renders the electric field inactive in the low-energy, long-wavelength regime. In contrast, there is no screening of the magnetic field, allowing it to exhibit its own dynamics even in this regime. Consequently, the magnetic field can be large and is treated as an $O(1)$ quantity, despite the fact that $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$ involves one spatial gradient. The presence of an $O(1)$ magnetic field breaks the $SO(3)$ symmetry in the constitutive relations for $\Theta^{\mu\nu}$, introducing anisotropy even in ideal hydrodynamics. Specifically, we can define a normalized vector $b^\mu = B^\mu/B$, where $B = \sqrt{-B^\mu B_\mu}$, satisfying $b^2 = -1$ and $b\cdot u = 0$, as an additional building block for hydrodynamic constitutive relations. For example, for a parity-even and charge neutral fluid, the energy–momentum tensor can be decomposed into

$$\Theta^{\mu\nu} = \epsilon u^\mu u^\nu - P_\perp\Xi^{\mu\nu} + P_\parallel b^\mu b^\nu + \Theta^{\mu\nu}_{(1)}, \tag{68}$$

where $\Xi^{\mu\nu} = \Delta^{\mu\nu} + b^\nu b^\nu$ is a projector transverse to both u^μ and b^μ. The terms $P_\perp$ and $P_\parallel$ represent the pressures in directions transverse and parallel to the magnetic field, respectively. Note that when we allow an environmental parity violation (e.g., when there is a density imbalance between the right- and left-hand particles in the fluid) and a finite charge density, the term $u^{(\mu}b^{\nu)}$ can appear in the zeroth order. The term $\Theta^{\mu\nu}_{(1)}$ (which is assumed to be symmetric because the spin degree of freedom is typically disregarded in MHD) denotes a collection of terms that are at least of order $O(\partial)$ in the gradient expansion and consistent with the Onsager relations. For a parity-even fluid, all such terms are expressed as $\Theta^{\mu\nu}_{(1)} = \sum_{i=1}^{7} \lambda_i \eta_i^{\mu\nu\rho\sigma} \nabla_\rho u_\sigma$, where λ_i is the corresponding transport coefficient [121–123]:

$$\eta_1^{\mu\nu\rho\sigma} = b^\mu b^\nu b^\rho b^\sigma, \tag{69a}$$

$$\eta_2^{\mu\nu\rho\sigma} = \Xi^{\mu\nu}\Xi^{\rho\sigma}, \tag{69b}$$

$$\eta_3^{\mu\nu\rho\sigma} = -\Xi^{\mu\nu} b^\rho b^\sigma - \Xi^{\rho\sigma} b^\mu b^\nu, \tag{69c}$$

$$\eta_4^{\mu\nu\rho\sigma} = -2\left[b^{(\mu}\Xi^{\nu)\rho} b^\sigma + b^{(\mu}\Xi^{\nu)\sigma} b^\rho\right], \tag{69d}$$

$$\eta_5^{\mu\nu\rho\sigma} = 2\,\Xi^{\rho(\mu}\Xi^{\nu)\sigma} - \Xi^{\mu\nu}\Xi^{\rho\sigma}, \tag{69e}$$

$$\eta_6^{\mu\nu\rho\sigma} = -b^{(\mu} b^{\nu)\rho} b^\sigma - b^{(\mu} b^{\nu)\sigma} b^\rho, \tag{69f}$$

$$\eta_7^{\mu\nu\rho\sigma} = \Xi^{\rho(\mu} b^{\nu)\sigma} + \Xi^{\sigma(\mu} b^{\nu)\rho}, \tag{69g}$$

where $b^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} u_\rho b_\sigma$ is a cross projector that appears only when charge-conjugation symmetry is violated (e.g., when a net charge density is presented).

Similar to the discussions above regarding MHD, we can consider a scenario for spin hydrodynamics where the vorticity is treated as zeroth-order in gradients, while the gradients of other thermodynamic quantities are treated as first-order. In line with the MHD, this framework has been referred to as gyrohydrodynamics in Ref. [72]. To simplify the notation, we reuse b^μ to denote the unit vector along the vorticity,

$$b^\mu = \varpi^\mu / \sqrt{-\varpi_\mu \varpi^\mu} = \omega^\mu / \sqrt{-\omega_\mu \omega^\mu}, \tag{70}$$

with $\varpi^\mu = -\epsilon^{\mu\nu\rho\sigma} u_\nu \partial_\rho \beta_\sigma / 2 = \beta\omega^\mu$ the thermal vorticity vector. We chose the pseudo-gauge such that the spin tensor is totally antisymmetric. Using u^μ, b^μ as well as $g^{\mu\nu}, \epsilon^{\mu\nu\rho\sigma}$ as building blocks, we can decompose $\Theta^{\mu\nu}$ and $\Sigma^{\mu\nu\rho}$ into the following irreducible forms:

$$\begin{aligned}\Theta^{\mu\nu} = {} & \varepsilon u^\mu u^\nu - P_\perp \Xi^{\mu\nu} + P_\parallel b^\mu b^\nu + P_\times b^{\mu\nu} \\ & + q^\mu u^\nu - u^\mu q^\nu + \Theta^{\mu\nu}_{s(1)} + \phi^{\mu\nu},\end{aligned} \tag{71}$$

$$\Sigma^{\mu\nu\rho} = -\epsilon^{\mu\nu\rho\lambda} S_\lambda = -\epsilon^{\mu\nu\rho\lambda}(n_5 u_\lambda - S_\parallel b_\lambda + S_{\perp\lambda}), \tag{72}$$

where $P_{\perp,\parallel,\times}$ represent pressures (which will be counted as $O(1)$ quantities in gradient expansion) in different directions, whose physical meaning will become clear shortly. The quantity $S_\parallel = b \cdot S$ denotes the spin component in the direction of the vorticity, while $S_\perp^\mu = \Xi^{\mu\nu} S_\nu$ denotes the spin component transverse to the vorticity. As before, we chose the Landau–Lifshitz frame, with q^μ, $\Theta^{\mu\nu}_{s(1)}$, $\phi^{\mu\nu}$, and $S_\perp^\mu$ transverse to u^μ. Note again that, with this fully antisymmetric choice of spin tensor, the q^μ vector is no longer independent, but is determined by S^μ through Eq.(64).

The power counting scheme is such that $S_\parallel$ is counted as order one, whereas $\phi^{\mu\nu} = -\phi^{\nu\mu}$, $S_\perp^\mu$, n_5, and q^μ (see Eq.(64)) are counted as at least $O(\partial)$. Additionally, we will count S^μ as $O(\hbar)$ (since spin is totally quantum in nature) in comparison with other thermodynamic quantities, which can appear even at the classical level and are therefore assigned $O(\hbar^0)$. This allows for a double expansion in both ∂ and $\hbar$. For the entropy current, we can write $s^\mu = su^\mu + s^\mu_{(1)}$ and use Eq.(33). It is straightforward to derive the divergence of the entropy current, and after some calculations, it was found that up to $O(\hbar\partial^2, \partial^3)$ [72]:

$$\begin{aligned}\partial_\mu s^\mu = {} & \left[s - \beta\left(\varepsilon + P_\perp\right)\right]\theta - (P_\parallel - P_\perp - \mu_\parallel S_\parallel) b^\mu b^\nu \partial_\mu \beta_\nu \\ & + P_\times b^{\mu\nu}(\partial_\mu \beta_\nu + \beta\mu_{\mu\nu}) + \Theta^{\mu\nu}_{s(1)} \partial_{(\mu}\beta_{\nu)} + \phi^{\mu\nu}(\partial_{[\mu}\beta_{\nu]} \\ & + \beta\mu_{\mu\nu}) + \partial_\mu\left(s^\mu_{(1)} - \beta\mu^\mu n_5\right) + O(\hbar\partial^2, \partial^3).\end{aligned} \tag{73}$$

The first line provides the zero-order contribution to the entropy production, which is expected to vanish so that they represent non-dissipative contributions. This gives

$$\varepsilon + P_\perp = Ts, \quad P_\parallel = P_\perp + \mu_\parallel S_\parallel, \quad P_\times = 0. \tag{74}$$

The first relation is the Gibbs–Duhem relation, indicating that $P_\perp$ can be interpreted as the thermodynamic pressure. The second relation shows that the pressure along the vorticity direction differs from the thermodynamic pressure by an amount due to spin polarization $\mu_\parallel S_\parallel$. This term is similar to the MB term in the magnetohydrodynamic constitutive relation. The third relation shows that there is no spin torque at the leading order.

At $O(\partial)$ order, the requirement of a semi-positive entropy production gives that

$$\Theta^{\mu\nu}_{s(1)} = T\eta^{\mu\nu\rho\sigma}\partial_{(\rho}\beta_{\sigma)} + T\xi^{\mu\nu\rho\sigma}(\partial_{[\rho}\beta_{\sigma]} + \beta\mu_{\rho\sigma}), \tag{75}$$

$$\phi^{\mu\nu} = T\gamma^{\mu\nu\rho\sigma}(\partial_{[\rho}\beta_{\sigma]} + \beta\mu_{\rho\sigma}) + T\xi'^{\,\mu\nu\rho\sigma}\partial_{(\rho}\beta_{\sigma)}, \tag{76}$$

$$s^\mu_{(1)} = \beta\mu^\mu n_5, \tag{77}$$

where $\eta^{\mu\nu\rho\sigma}$ and $\gamma^{\mu\nu\rho\sigma}$ are the usual and rotational viscous tensors representing the response of the symmetric and antisymmetric parts of the energy–momentum tensor to fluid shear and expansion, and the difference between vorticity and spin potential, respectively, and $\xi^{\mu\nu\rho\sigma}$ and $\xi'^{\,\mu\nu\rho\sigma}$ are two cross viscous tensors. Note that the cross viscous tensors are not independent of each other but inter-related according to Onsager's reciprocal principle, $\xi'^{\,\mu\nu\rho\sigma}(b) = \xi^{\rho\sigma\mu\nu}(-b)$. By decomposing these tensors into irreducible structures, one obtains a number of new transport coefficients (viscosities) that characterize the response of the fluid to gradients of fluid velocity and spin potential [72]:

$$\begin{aligned}\eta^{\mu\nu\rho\sigma} &= \zeta_\perp \Xi^{\mu\nu}\Xi^{\rho\sigma} + \zeta_\parallel b^\mu b^\nu b^\rho b^\sigma + \zeta_\times\left(b^\mu b^\nu \Xi^{\rho\sigma} + \Xi^{\mu\nu} b^\rho b^\sigma\right)\\ &+ \eta_\perp\left(\Xi^{\mu\rho}\Xi^{\nu\sigma} + \Xi^{\mu\sigma}\Xi^{\nu\rho} - \Xi^{\mu\nu}\Xi^{\rho\sigma}\right)\\ &+ 2\eta_\parallel\left(b^\mu \Xi^{\nu(\rho}b^{\sigma)} + b^\nu \Xi^{\mu(\rho}b^{\sigma)}\right)\\ &+ 2\eta_{H_\perp}\left(\Xi^{\mu(\rho}b^{\sigma)\nu} + \Xi^{\nu(\rho}b^{\sigma)\mu}\right)\\ &+ 2\eta_{H_\parallel}\left(b^\mu b^{\nu(\rho}b^{\sigma)} + b^\nu b^{\mu(\rho}b^{\sigma)}\right),\end{aligned} \tag{78}$$

$$\begin{aligned}\gamma^{\mu\nu\rho\sigma} &= \gamma_\perp\left(\Xi^{\mu\rho}\Xi^{\nu\sigma} - \Xi^{\mu\sigma}\Xi^{\nu\rho}\right)\\ &+ 2\gamma_\parallel\left(b^\mu \Xi^{\nu[\rho}b^{\sigma]} - b^\nu \Xi^{\mu[\rho}b^{\sigma]}\right)\\ &+ 2\gamma_H\left(b^\mu b^{\nu[\rho}b^{\sigma]} - b^\nu b^{\mu[\rho}b^{\sigma]}\right),\end{aligned} \tag{79}$$

$$\begin{aligned}\xi^{\mu\nu\rho\sigma} &= 2\xi_\parallel\left(b^\mu \Xi^{\nu[\rho}b^{\sigma]} + b^\nu \Xi^{\mu[\rho}b^{\sigma]}\right)\\ &+ \zeta_{H_\perp}\Xi^{\mu\nu}b^{\rho\sigma} + \zeta_{H\parallel} b^\mu b^\nu b^{\rho\sigma}\\ &+ 2\xi_H\left(b^\mu b^{\nu[\rho}b^{\sigma]} + b^\nu b^{\mu[\rho}b^{\sigma]}\right),\end{aligned} \tag{80}$$

where the η's, ζ's, γ's, and ξ's are transport coefficients. Especially, those with subscript "H" are Hall-type transport coefficients which do not contribute to the entropy production and thus their sign are not constrained by the second law of local thermodynamics. One may wonder why the term $\propto b^{\mu\nu}b^{\rho\sigma}$ (such term would contribute to an $O(\partial)$ analog of $P_\times$ term in Eq.(71)) does not appear in $\gamma^{\mu\nu\rho\sigma}$. This is because it is not independent of the other terms in $\gamma^{\mu\nu\rho\sigma}$ [121]. Note that the expression for $\xi^{\mu\nu\rho\sigma}$ is different from that in Ref. [72] but equivalently yields the same constitutive relations once substituted into Eq.(75).

4.3 A spin Cooper–Frye formula

To apply spin hydrodynamics to specific physical systems, we need to know the appropriate observables for the detection of spin degrees of freedom in the fluid. In principle, the presence of the spin degree of freedom in the fluid should modify the usual hydrodynamic quantities such as the energy density and fluid velocity, but when the spin density is not large (nevertheless it is always suppressed by $\hbar$ comparing to the traditional hydrodynamic quantities), such modification is small. In heavy ion collisions, the natural observable is the spin polarization of hadrons, including spin-1/2 hyperons and spin-1 vector mesons. Hyperons are of special interest because they primarily decay via weak interactions such that the momentum of one of the daughter particles tends to align with the spin direction of the hyperon. To obtain the spin polarization observables of a hadron from spin hydrodynamics, machinery is required to convert the outcomes of spin hydrodynamics, such as fluid velocity, temperature, and spin potential, to measurable hadronic observables.

In the application of traditional hydrodynamics to heavy-ion collisions, the hadron momentum spectra are typically obtained using the Cooper–Frye formula:

$$E_p\frac{dN_i}{d\boldsymbol{p}^3} = \int_\Xi d\Xi_\mu(x)p^\mu f_i(x,p), \tag{81}$$

where the integral is over the freeze-out hypersurface (where particlization occurs) Ξ, and $f_i(x,p)$ is the distribution function of species i of the hadrons in the fluid. Any possible degeneracy of the hadrons should be accounted for in f_i. For example, when dissipative effects are neglected, the distribution function f_i is typically taken as the Fermi-Dirac or Bose–Einstein functions $f_{F,B}(p\cdot\beta-\mu_i)$ with μ_i is the chemical potential. The above Cooper–Frye formula has been widely used in hydrodynamic simulations in heavy-ion collisions and has proven to be very successful. Therefore, to extend traditional hydrodynamics to spin hydrodynamics, we also need to generalize the above Cooper–Frye formula to a spin Cooper–Frye formula.

Let us consider a system in which thermal equilibrium is reached locally but not necessarily globally. The density operator $\hat\rho$ for the description of such an ensemble is obtained by maximizing the entropy functional under the constraints of the given energy-momentum and angular momentum (or spin) densities:

$$\begin{aligned}S[\hat\rho] =& -\mathrm{Tr}(\hat\rho\ln\hat\rho) + \lambda(\mathrm{Tr}\hat\rho - 1) - \int_\Xi d\Xi_\mu\left[\mathrm{Tr}(\hat\rho\hat\Theta^{\mu\nu}) - \Theta^{\mu\nu}\right]\beta_\nu\\ &+ \frac{1}{2}\int_\Xi d\Xi_\mu\left[\mathrm{Tr}(\hat\rho\hat\Sigma^{\mu\nu\rho}) - \Sigma^{\mu\nu\rho}\right]\mu_{\nu\rho},\end{aligned} \tag{82}$$

where $\Theta^{\mu\nu}(x)$ and $\Sigma^{\mu\nu\rho}(x)$ are the actual local energy–momentum tensor and spin tensor, respectively, and $\beta_\nu(x)$ and $\mu_{\nu\rho}(x)$ are the corresponding Lagrange multipliers. The Lagrange multiplier λ is introduced to normalize $\hat\rho$ and is related to the partition function Z as $\exp(1-\lambda) = Z$. The resultant density operator is the local-equilibrium density operator [124–127]:

$$\hat\rho_{\mathrm{LE}} = \frac{1}{Z_{\mathrm{LE}}}\exp\left\{-\int_\Xi d\Xi_\mu(x)\left[\hat\Theta^{\mu\nu}(x)\beta_\nu(x) - \frac{1}{2}\hat\Sigma^{\mu\rho\sigma}(x)\mu_{\rho\sigma}(x)\right]\right\}, \tag{83}$$

where Z_{LE} denote the local-equilibrium partition function. Now, we see that $\hat\rho_{\mathrm{LE}}$ is determined by the local thermodynamic quantities β^μ and $\mu_{\rho\sigma}$. If we calculate the spin density $\Sigma^{\mu\rho\sigma}(x)$ using this density operator, we obtain a relationship between $\Sigma^{\mu\rho\sigma}(x)$ and the local thermodynamic quantities (and possibly their derivatives). However, this is not particularly useful in the context of heavy-ion collisions because what is measured is the spin density in momentum space, rather than the coordinate space. To express such a relation for the spin density in momentum space or phase space, the most natural approach is to use the Wigner function.

To illustrate how this can be achieved, we consider a Dirac fermion system as an example. The Wigner operator is defined as follows:

$$\hat{W}(x,p) = \int d^4 s e^{-ip\cdot s} \bar{\hat{\psi}}\left(x + \frac{s}{2}\right) \otimes \hat{\psi}\left(x - \frac{s}{2}\right), \tag{84}$$

where $[\bar{\hat{\psi}} \otimes \hat{\psi}]_{ab} \equiv \bar{\hat{\psi}}_b \hat{\psi}_a$ with a, b spinor indices. We choose the canonical pseudo-gauge, in which the energy–momentum tensor operator and spin tensor operator are given by

$$\hat{\Theta}^{\mu\nu} = \bar{\hat{\psi}} i\gamma^\mu \partial^\nu \hat{\psi} - \eta^{\mu\nu}\hat{\mathcal{L}}, \tag{85}$$

$$\hat{\Sigma}^{\mu\nu\rho} = \frac{1}{4}\bar{\hat{\psi}}\{\gamma^\mu, \sigma^{\rho\sigma}\}\hat{\psi} = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\hat{\psi}}\gamma_\sigma\gamma_5\hat{\psi}. \tag{86}$$

where $\hat{\mathcal{L}}$ is the Lagrangian (in the following, we consider free fermions, so that $\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi$ is in quadratic form, and the second term in $\hat{\Theta}^{\mu\nu}$ vanishes when using the equation of motion of the field operator) and $\sigma^{\rho\sigma} = i[\gamma^\rho, \gamma^\sigma]/2$. Note that the second equation indicates that the spin vector $\hat{S}_\sigma = (1/2)\bar{\hat{\psi}}\gamma_\sigma\gamma_5\hat{\psi}$ is half the axial current. Both $\hat{\Theta}^{\mu\nu}(x)$ and $\hat{\Sigma}^{\mu\nu\rho}(x)$ are local Heisenberg operators. We can extend them into operators in the phase space by using the Wigner transformation, for example,

$$\begin{aligned}\hat{\Sigma}^{\mu\nu\rho}(x,p) =& -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \int d^4 s e^{-ip\cdot s}\bar{\hat{\psi}}\left(x + \frac{s}{2}\right)\gamma_\sigma\gamma_5\hat{\psi}\left(x - \frac{s}{2}\right) \\ =& -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}_{\mathrm{D}}\left[\gamma_\sigma\gamma_5\hat{W}(x,p)\right],\end{aligned} \tag{87}$$

where Tr_{D} is the trace over the Dirac space. It is easy to see that $\int d^4p/(2\pi)^4\hat{\Sigma}^{\mu\nu\rho}(x,p) = \hat{\Sigma}^{\mu\nu\rho}(x)$. The integration of $\hat{\Sigma}^{\mu\nu\rho}(x,p)$ over a certain spacelike hypersurface gives us the spin tensor in momentum space (whose exact meaning will be clarified later), whose ensemble average under $\hat{\rho}_{\mathrm{LE}}$ is exactly the quantity that we are looking for. Therefore, we must calculate the Wigner function under local equilibrium:

$$W(x,p) = \langle\hat{W}(x,p)\rangle = \mathrm{Tr}\left[\hat{\rho}_{\mathrm{LE}}\hat{W}(x,p)\right], \tag{88}$$

where Tr denotes the trace over a complete set of microstates in the system. To proceed, the local-equilibrium density operator can be rewritten as $\hat{\rho}_{\mathrm{LE}} = \exp(\hat{A} + \hat{B})/Z_{\mathrm{LE}}$ with the abbreviations

$$\hat{A} = -\hat{P}^\mu\beta_\mu(x), \tag{89}$$

$$\hat{B} = -\int d\Xi_\mu(y)\left[\hat{\Theta}^{\mu\nu}(y)\Delta\beta_\nu(y) - \frac{1}{2}\hat{\Sigma}^{\mu\rho\sigma}(y)\mu_{\rho\sigma}(y)\right], \tag{90}$$

where $\hat{P}^\mu = \int d\Xi_\nu(y)\hat{\Theta}^{\nu\mu}(y)$, $\Delta\beta_\mu(y) = \beta_\mu(y) - \beta_\mu(x)$. The purpose of rewriting $\hat{\rho}_{\mathrm{LE}}$ in this form is that, the correlation length between the spin tensor and the energy–momentum tensor is typically small. Within this correlation length, we can assume that local thermodynamic quantities, such as β_μ, vary only slightly. Given that $\mu_{\rho\sigma}$ is also small at the hypersurface Ξ (which is a reasonable assumption for heavy-ion collisions, although it may not hold for a strongly polarized medium), we assign $\Delta\beta_\nu \sim \mu_{\rho\sigma} \sim O(\partial)$, therefore, $\hat{A} \sim O(1)$, $\hat{B} \sim O(\partial)$. Using this power-counting scheme, we can expand the right-hand side of Eq.(88) order by order in ∂ by applying the identity $e^{\hat{A}+\hat{B}} = e^{\hat{A}} + e^{\hat{A}}\int_0^1 d\lambda e^{-\lambda\hat{A}}\hat{B}e^{\lambda\hat{A}} + \cdots$, and obtain

$$W(x,p) = W_0(x,p) + W_1(x,p) + \cdots, \tag{91}$$

where

$$W_0(x,p) = \langle\hat{W}(x,p)\rangle_0 \equiv \frac{1}{Z_0}\mathrm{Tr}\left(e^{\hat{A}}\hat{W}(x,p)\right), \tag{92}$$

$$W_1(x,p) \equiv \langle\hat{W}(x,p)\rangle_{(\Theta)} + \langle\hat{W}(x,p)\rangle_{(\Sigma)}, \tag{93}$$

with

$$\begin{aligned}\langle\hat{W}(x,p)\rangle_{(\Theta)} &\equiv -\int_0^1 d\lambda\int d\Xi_\rho(y)\Delta\beta_\nu(y)\langle\hat{\Theta}^{\rho\nu}(y - i\lambda\beta(x))\hat{W}(x,p)\rangle_{0,c}, \\ \langle\hat{W}(x,p)\rangle_{(\Sigma)} &\equiv \frac{1}{2}\int_0^1 d\lambda\int d\Xi_\nu(y)\mu_{\rho\sigma}(y)\langle\hat{\Sigma}^{\nu\rho\sigma}(y - i\lambda\beta(x))\hat{W}(x,p)\rangle_{0,c},\end{aligned} \tag{94}$$

and $Z_0 = \mathrm{Tr}e^{\hat{A}}$. Here, $\langle\cdots\rangle_{0,c}$ means the connected part of the correlation. The calculation then will depend on the shape of the hypersurface Ξ. For illustration, we consider Ξ to be the 3-space at some time t so that its normal direction is $\hat{t}^\mu = (1, \mathbf{0})$. The calculation is then straightforward using the free field operator

$$\hat{\psi}(x) = \sum_{\sigma=1}^{2}\frac{1}{(2\pi)^{3/2}}\int\frac{d^3\boldsymbol{k}}{2E_k}\left[u_\sigma(\boldsymbol{k})e^{-ik\cdot x}\hat{a}_\sigma(\boldsymbol{k}) + v_\sigma(\boldsymbol{k})e^{ik\cdot x}\hat{b}_\sigma^\dagger(\boldsymbol{k})\right], \tag{95}$$

where $E_k = \sqrt{\boldsymbol{k}^2 + m^2}$ and $\hat{a}_\sigma(\boldsymbol{k}), \hat{b}_\sigma(\boldsymbol{k})$ are annihilation operators for particles and antiparticles satisfying the anti-commutation relation $\{\hat{a}_\sigma(\boldsymbol{k}), \hat{a}_{\sigma'}^\dagger(\boldsymbol{q})\} = \{\hat{b}_\sigma(\boldsymbol{k}), \hat{b}_{\sigma'}^\dagger(\boldsymbol{q})\} = 2E_k\delta_{\sigma\sigma'}\delta^3(\boldsymbol{k} - \boldsymbol{q})$ and the relation $\langle\hat{a}_\sigma^\dagger(\boldsymbol{k})\hat{a}_{\sigma'}(\boldsymbol{q})\rangle_0 = \langle\hat{b}_\sigma^\dagger(\boldsymbol{k})\hat{b}_{\sigma'}(\boldsymbol{q})\rangle_0 = 2E_k\delta_{\sigma\sigma'}\delta^3(\boldsymbol{k} - \boldsymbol{q})n_F(k\cdot\beta)$. In the following, we consider only the particle branch; the antiparticle branch is completely similar. The zeroth-order Wigner function can be easily obtained: $W_0(x,p) = 2\pi(\not{p} + m)\theta(p_0)\delta(p^2 - m^2)n_F(p\cdot\beta)$, which is spin independent: $\mathrm{Tr}_D\left[\gamma^\mu\gamma_5 W_0(x,p)\right] = 0$.

The first-order Wigner function reads

$$\begin{aligned}\langle\hat{W}(x,p)\rangle_{(\Theta/\Sigma)} =& 2\pi\int_0^1 d\lambda\int\frac{d^3\boldsymbol{k}}{2E_k}\int\frac{d^3\boldsymbol{q}}{2E_q}\delta^4\left(p - \frac{q+k}{2}\right) \\ &(\gamma\cdot k + m)\hat{t}_\mu I^\mu_{(\Theta/\Sigma)}(\gamma\cdot q + m) \times e^{\lambda(k-q)\cdot\beta(x)}n_F(k)\left[1 - n_F(q)\right],\end{aligned} \tag{96}$$

where $n_F(p) = n_F[\beta(x)\cdot p]$, $I^\mu_{(\Theta)} = -\gamma^\mu p^\nu\left[\partial_\lambda\beta_\nu(x)\right]\Delta^\lambda_\beta\left[i\partial_q^\beta\delta^3(\boldsymbol{q} - \boldsymbol{k})\right]$, and $I^\mu_{(\Sigma)} = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\gamma^5\gamma_\nu\mu_{\rho\sigma}\delta^3(\boldsymbol{q} - \boldsymbol{k})$ with $\Delta^{\mu\nu} = \eta^{\mu\nu} - \hat{t}^\mu\hat{t}^\nu$. To obtain this result, we have used

$$\int d\Xi_\mu(y)(y-x)^\alpha e^{-i(p-q)\cdot(y-x)} = (2\pi)^3 \hat{t}_\mu \Delta^\alpha_\beta \frac{i\partial}{\partial p_\beta}\delta^3(\boldsymbol{p}-\boldsymbol{q}), \tag{97}$$

which is valid when Ξ is a 3-space. In heavy-ion collisions, the true freeze-out hypersurface Ξ is of course not a 3-space and thus correction due to the non-flatness of Ξ would appear; see discussions in Refs. [128, 129].

Using the first-order Wigner function in Eq. (96), the local-equilibrium spin vector in the phase space is directly obtained by finishing the trace over the Dirac space [130, 131]:

$$S_\mu(x,p) = -4\pi\delta(p^2-m^2)\theta(p_0)n_F(p)[1-n_F(p)] \left\{\frac{1}{4}\epsilon_{\mu\nu\rho\sigma}p^\nu\mu^{\rho\sigma} + \Sigma^{\hat{t}}_{\mu\nu}\left[(\xi^{\nu\lambda}+\Delta\mu^{\nu\lambda})p_\lambda\right]\right\}, \tag{98}$$

where $\Sigma^{\hat{t}}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}p^\rho\hat{t}^\sigma/(2p\cdot\hat{t})$, $\xi_{\mu\nu} = \partial_{(\mu}\beta_{\nu)}$ is the thermal shear tensor, and $\Delta\mu^{\mu\nu} = \mu^{\mu\nu} - \varpi^{\mu\nu}$ is the difference between the spin potential and the thermal vorticity tensor.

With this spin vector in phase space, the spin vector per particle in momentum space is obtained by average over hypersurface Ξ [130, 131]:

$$\begin{aligned} S_\mu(p) &= \frac{1}{2}\frac{\int d\Xi(x)\cdot p\mathrm{Tr}_D[\gamma^\mu\gamma^5 W(x,p)]}{\int d\Xi\cdot p\mathrm{Tr}_D[W(x,p)]} \\ &= -\frac{\int d\Xi\cdot p\left\{\epsilon_{\mu\nu\alpha\beta}p^\nu\mu^{\alpha\beta} + 4\Sigma^{\hat{t}}_{\mu\nu}\left[p_\lambda(\xi^{\nu\lambda}+\Delta\mu^{\nu\lambda})\right]\right\}n_F(1-n_F)}{8m\int d\Xi\cdot p\, n_F}, \end{aligned} \tag{99}$$

where p^μ on the right-hand side is on-shell. This is a Cooper-Frye-type formula for the spin vector, which connects the momentum-space distribution of the mean spin vector of particles emitted from Ξ with the fluid properties characterized by $\mu^{\mu\nu}(x)$ and $\beta^\mu(x)$ on Ξ. Thus, once these fluid variables are obtained from spin hydrodynamics, this spin Cooper–Frye formula allows us to convert them into the mean spin vector in momentum space, which is a directly measurable quantity.

We provide several comments before concluding this subsection. First, at local equilibrium, the thermal shear tensor can induce spin polarization, which has important implications for the spin polarization phenomena in heavy-ion collisions [128, 132–134]. Second, when the system is in global equilibrium, the spin potential is determined by the thermal vorticity and the thermal shear tensor $\xi_{\mu\nu}$ vanishes. In this case, the spin Cooper–Frye formula is reduced to that obtained in Refs. [20–22]. Third, we did not include the effects of finite baryon chemical potential. Its inclusion is straightforward, with the modification that the distribution function $n_F(p\cdot\beta)\to n_F(p\cdot\beta-\alpha)$, where $\alpha=\mu/T$. Additionally, a new term $4\int d\Xi\cdot p\Sigma^{\hat{t}}_{\mu\nu}\partial^\nu\alpha$ should be added to the numerator of Eq.(99), which is referred to as the spin Hall effect [135]. Fourth, formula (99) depends on the choice of pseudo-gauge [130, 131]. In particular, it is possible to completely eliminate the contributions of thermal shear by adopting appropriate pseudo-gauges. Therefore, when applying this formula to spin hydrodynamics, it is important to carefully choose a pseudo-gauge to maintain consistency.

5 Summary and outlooks

This article provides a pedagogical introduction to relativistic spin hydrodynamics. First, we demonstrate how one can derive a set of hydrodynamic equations from conservation equations based on the requirements of local thermodynamic laws, primarily the second law of thermodynamics. We then extended this framework to include the conservation of angular momentum, which leads to spin hydrodynamics. In the framework of spin hydrodynamics, the new (quasi-)hydrodynamic variable is spin density. Owing to spin-orbit coupling, the spin density is not a strict hydrodynamic variable but rather a quasi-hydrodynamic variable. It relaxes to a local equilibrium value determined by the local thermal vorticity through dissipative conversion of the spin and orbital angular momenta. We demonstrate how such dissipative processes are characterized by two new transport coefficients: one for boosting heat conductivity and the other for rotational viscosity.

We discuss several interesting aspects of spin hydrodynamics. First, we address the pseudo-gauge ambiguity in defining the spin tensor, which reflects the freedom to separate the total angular momentum into spin and orbital components. One consequence of this pseudo-gauge ambiguity is that we have the flexibility to choose spin tensors with different symmetries in their indices as the starting point for the derivation of spin hydrodynamics, leading to different constitutive relations. Second, we emphasize the importance of derivative power counting in the formulation of spin hydrodynamics. In particular, for a strongly vortical (or strongly spin-polarized) fluid, it is natural to assign the vorticity and spin potential as being of similar strength to other local thermodynamic quantities, such as temperature, in terms of derivative powers. This is analogous to the magnetohydrodynamics. As a result, anisotropy emerges in the constitutive relations both at the zeroth order and the first order in derivatives. This framework is well-suited for describing strongly vortical or spin-polarized fluids. Third, for potential applications of spin hydrodynamics, such as in heavy-ion collisions, we require a method to convert the results of spin hydrodynamics—specifically, the spin density (or spin potential), temperature, and fluid velocity—into momentum-space observables. To this end, we give a spin Cooper-Frye formula for Dirac fermions, and a similar formula can also be derived for spin-one vector bosons.

Spin hydrodynamics is an area of intensive study with many interesting aspects already explored and many more awaiting investigations. We provide a brief discussion of some of these topics.

(1) Spin magnetohydrodynamics. When the constituents of the fluid are charged, the fluid can interact with the electromagnetic fields and behave like a magnetized fluid. In this case, it is convenient to extend spin hydrodynamics to spin magnetohydrodynamics [136–140]. As electric fields are easily screened, they are not typically described as hydrodynamic variables. Therefore, the new hydrodynamic variable is the magnetic field (more precisely, the magnetic flux), $B^\mu = \tilde{F}^{\mu\nu}u_\nu$, which is counted as an $O(1)$ quantity in derivative power counting. The conservation law is simply a Bianchi identity.

$$\partial_\mu \tilde{F}^{\mu\nu} = 0. \tag{100}$$

Here, $\tilde{F}^{\mu\nu} = (1/2)\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the dual Maxwell tensor. This equation should be combined with the conservation laws of energy–momentum and angular momentum to form complete equations of motion for the fluid. Expanding $\tilde{F}^{\mu\nu}$ in terms of hydrodynamic variables yields [120]:

$$\tilde{F}^{\mu\nu} = B^\mu u^\nu - B^\nu u^\mu + \tilde{F}^{\mu\nu}_{(1)}, \tag{101}$$

where $\tilde{F}^{\mu\nu}_{(1)}$ and B^μ are transverse to u^μ. Local thermodynamic laws can be imposed, for example, the first law and a generalized Gibbs–Duhem relation, as follows:

$$Tds + \frac{1}{2}\mu_{\mu\nu}dS^{\mu\nu} + H_\mu dB^\mu = d\varepsilon, \tag{102}$$

$$Ts + \frac{1}{2}\mu_{\mu\nu}S^{\mu\nu} + H_\mu B^\mu = \varepsilon + P, \tag{103}$$

with H_μ the "magnetic potential" conjugate to the magnetic flux (physically, it can be interpreted as the in-medium magnetic field strength). The covariant form for the Gibbs–Duhem relation is

$$s^\mu = P\beta^\mu + \Theta^{\mu\nu}\beta_\nu - \frac{1}{2}\Sigma^{\mu\rho\sigma}\alpha_{\rho\sigma} + \tilde{F}^{\mu\nu}\gamma_\nu, \tag{104}$$

with $\gamma^\mu = \beta H^\mu$. The second law of thermodynamics requires $\partial_\mu s^\mu \geq 0$, which imposes constraints on the possible forms of the constitutive relations order by order in the gradient expansion. Recently, such a framework for spin magnetohydrodynamics was discussed (see Refs. [139, 140] for further detail).

It would be interesting to extend these studies to include possible parity-violating effects, thereby obtaining spin magnetohydrodynamics in a chiral conducting medium. This provides a bridge between spin magnetohydrodynamics and chiral magnetohydrodynamics. Another issue that may affect the formulation of spin magnetohydrodynamics is pseudo-gauge ambiguity. As we have seen, such an ambiguity is crucial for the formulation of spin hydrodynamics, and it would be interesting to explore how it influences the formulation of spin magnetohydrodynamics. Finally, exploring possible collective modes and instabilities in such a fluid is also important. This would be valuable for potential applications (e.g., possible dynamo mechanisms owing to spin degrees of freedom) in what we might call spin plasma, whether in heavy-ion collisions or astrophysical systems.

(2) Calculation of the new transport coefficients. As we have seen, new transport coefficients appear in spin hydrodynamics, most notably rotational viscosity η_s. Strictly speaking, η_s, unlike the typical shear viscosity η, is not a transport coefficient in the traditional sense. It does not characterize the ability to transport spin within the fluid; rather, it represents how quickly the spin density relaxes to its equilibrium value, which is determined by thermal vorticity. This can be easily understood by rewriting Eq.(8) in the canonical pseudo-gauge and in component form (keeping linear terms in spin density and velocity): $\partial_t S^i \approx -\eta_s(\mu^i - \varpi^i)$ where $\mu^i = \epsilon^{ijk}\mu_{ik}$, which leads to $\partial_t \mu^i = -\Gamma_s(\mu^i - \varpi^i)$ with $\Gamma_s = \eta_s/\chi_s$ the spin relaxation rate and χ_s the spin susceptibility. Nevertheless, the calculation of Γ_s and η_s is important for understanding the evolution of spin polarization. Recently, Γ_s has been computed perturbatively for heavy quarks in hot QCD plasma [65, 67] and baryons in hot hadronic plasma [66]. Kinetic theory-based calculations have also been reported [67]. The results show that, for heavy quarks, this parameter can be parametrically small, making the spin degree of freedom a quasi-hydrodynamic mode. In future, the calculation of other new transport coefficients, such as those arising in gyrohydrodynamics [72], could also be crucial for understanding spin dynamics in different fluids. In addition, it is important to examine and understand the pseudo-gauge dependence of these new transport coefficients.

(3) Simulation of spin hydrodynamics. It is important to develop a suitable numerical framework for performing simulations to apply spin hydrodynamics to heavy-ion collisions. First-order relativistic hydrodynamic equations are known to suffer from numerical instabilities and emergence of acausal modes. The origin of this problem lies in the fact that first-order constitutive relations are non-dynamical, meaning that the response of the fluid to thermodynamic forces is instantaneous. One solution to this problem is to make the constitutive relations more dynamic. For example, the constitutive relation for the shear channel can be modified as

$$\tau_\pi (D\pi)^{\mu\nu} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}, \tag{105}$$

where $\pi^{\mu\nu}$ is the traceless symmetric part of $\Theta^{\mu\nu}_{(1)}$, $(D\pi)^{\mu\nu} \equiv (1/2)[\Delta^{\mu\rho}\Delta^{\nu\sigma} + \Delta^{\mu\sigma}\Delta^{\nu\rho} - (2/3)\Delta^{\mu\nu}\Delta^{\rho\sigma}]D\pi_{\rho\sigma}$ is the traceless part of the co-moving time derivative of $\pi_{\mu\nu}$, and τ_π represents how quickly $\pi^{\mu\nu}$ relaxes into the hydrodynamic constitutive relation. (Note that this procedure introduces a new dynamic mode that is not a hydrodynamic mode and relaxes on a timescale given by τ_π.) The use of such a modification has been successful in the numerical simulation of relativistic hydrodynamics. For relativistic spin hydrodynamics, modifications similar to the constitutive relations may be adopted to implement numerical simulations. This has recently been discussed in Refs. [41, 74, 92, 93, 95, 141, 142]. Essentially, the constitutive relation Eq. (41) is replaced by a dynamic relation

$$\tau_\phi (D\phi)^{\mu\nu} + \phi^{\mu\nu} = \eta_s \Delta^{\mu\rho}\Delta^{\nu\sigma}\left(\mu_{\rho\sigma} - T\varpi_{\rho\sigma}\right), \tag{106}$$

where τ_ϕ is the relaxation time for the antisymmetric part of the energy–momentum tensor and $(D\phi)^{\mu\nu} \equiv \Delta^{\mu\rho}\Delta^{\nu\sigma}D\phi_{\rho\sigma}$ is the transverse part of the co-moving time derivative of $\phi_{\mu\nu}$. With these modifications, a numerical simulation of relativistic spin hydrodynamics can be performed, which will provide valuable insights into spin polarization phenomena (see the recent progress in Refs. [143, 144]) such as those observed in heavy-ion collisions.

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