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Analytical expression for betatron tune shift due to field errors in lump quadrupole

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Abstract The integrated gradient of a quadrupole will be deviated by a close neighboring sextupole, and this is called the effect of fringe field interference. Using the Lie algebra techniques, an analytical expression for the betatron tune shift due to this effect has been derived. The process does not depend on the supposition of the thin-lens quadrupoles. It can be used to estimate the tune shift differences between the designed lattice and the one including the fringe field interference. More generally, the method can be applied to other kinds of fringe field interference.

Keywords Fringe field interference · Lie algebra technique · Betatron tune shift

1 Introduction

BEPC-II was upgraded from the previous Beijing Electron Positron Collider (BEPC), and its design luminosity is 1×10^{33} cm^{-2} s^{-1} at 1.89 GeV. Since BEPC-II uses the same tunnel as the previous BEPC [1], the longitudinal space occupied by the quads and sexts in four arcs is highly squeezed. As a result, the fringe field interferences are inevitable. We have already investigated the relative effects [2], including the changes in the integrated gradients and betatron tune shifts due to field interferences via 3D multipole expansion techniques and beam tracking, respectively. The 3D simulation model of a quad-sext assembly is shown in Fig. 1. For the BEPC-II case, the results show that the design value of the integrated gradient is reduced by 0.57 % due to the neighboring sext. The changes in horizontal tune and vertical tune are −0.0266 and −0.0335, respectively. According to the theory of beam–beam interactions, to arrive at higher luminosity, the tunes should be very close to a half-integer, so optimizing transverse tune is one of the most important issues at the commissioning stage, which needs us to control the betatron tune shift as accurately as possible. The betatron tune shift due to a distributed gradient error has long been established [3]. It involves an integral expression, and the rapid estimation of the tune shift needs a β function to be chosen at the quad center. For a thick-lens quad, this approximation would not be accurate enough. Therefore, it is necessary to modify the classical tune shift formula to a higher precision.

Section 2 presents the detailed process of derivation of the betatron tune shift using Lie algebra techniques. In Sect. 3, the tune shifts, based on the models of both thick-lens quad and thin-lens quad, are compared with the beam tracking results via Methodical Accelerator Design (MAD) [4]. Section 4 gives a brief summary.

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2 Derivation to betatron tune shift

For a quad with a distributed gradient error, the betatron tune shift is [3]

$$\Delta\mu_{x,y} = \frac{1}{4\pi}\int \beta_{x,y}(s)\Delta K_2(s)\mathrm{d}s, \tag{1}$$

where $\beta_{x,y}$ is the Twiss function in a quad and ΔK_2 is the strength error of a quad. If we want to make a rapid estimation, the β function will be substituted as a constant, usually the value at the quad center. In other words, the

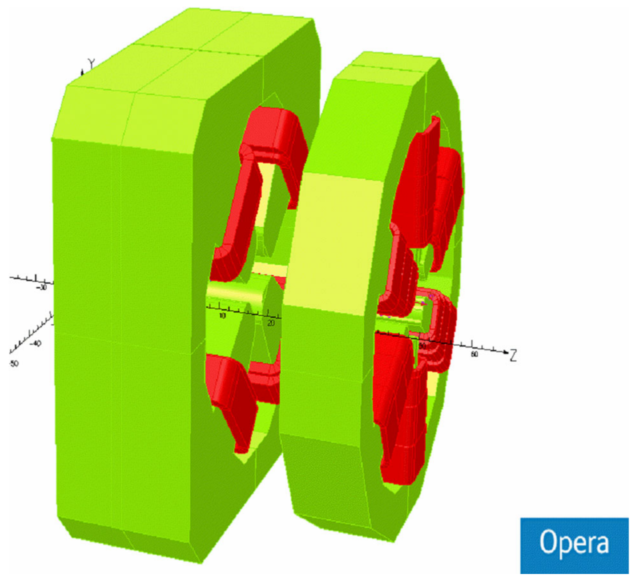

Fig. 1 Simulation model for BEPC-II ring quad–sext assembly (from *left* to *right*: quad magnet, sext magnet)

thin-lens quad is adopted. For a quad with a finite length, in which the β function changes sharply, e.g., the quads in the interaction region, the thin-lens quad is no longer accurate. The method used in this section gives the betatron tune shift of a thick-lens quad with a gradient error, and the results can be applied to the rapid tune shift estimations.

The analytical approach followed here is inspired by Irwin and Wang's paper [5]. They proposed a Lie algebra technique to obtain the explicit soft fringe maps of a quad. In their calculations, the Hamiltonian was sliced into many thin pieces and then these thin pieces were concatenated to form a thick quad. By successive application of the similarity transformation and BCH formula, the soft fringe maps were constructed after performing the integral over the whole quad. The normal quad Hamiltonian with a small δ perturbation for the on-momentum particle has the form

$$\begin{aligned} H_{\text{Quad}} &= \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{K_2}{2}(1-\delta)\left(x^2 - y^2\right) \\ &= H_2 - \frac{K_2\delta}{2}\left(x^2 - y^2\right) = H_2 + H_\delta \end{aligned}, \tag{2}$$

where x, y are the canonical coordinates, p_x, p_y are the canonical particle momenta, and δ is the reduction of the integrated gradient due to the effects of fringe field interference. The Hamiltonian is separated into two parts. H_δ represents the perturbed Hamiltonian with a small quantity and H_2 denotes a perfect quad. Adopting the procedures mentioned above, the map of a perturbed quad can be written as the product

$$\mathfrak{M} = M_{l/2} e^{-:F_\delta:} M_{l/2}, \tag{3}$$

where $M_{l/2}$ is the linear map across half the length of the thick quad and F_δ is the Lie generator of the exponential transformation

$$\begin{aligned} F_\delta &= \int_{-l/2}^{l/2} H_\delta(x(s), y(s))ds \\ &\quad - \frac{1}{2}\int_{-l/2}^{l/2} \mathrm{d}s \int_s^{l/2} [H_\delta(x(s), y(s)), H_\delta(x(s'), y(s'))]\mathrm{d}s' \ldots, \\ &= F_1 + F_2 + \cdots, \end{aligned} \tag{4}$$

where F_δ is the total kicks from the thick quad, in which F_1 and F_2 represent the first and the second kick effects, respectively. The latter is also known as the effects of kick on kick. In BEPC-II case, the second integral, F_2, is smaller than the first integral, F_1, by a factor δ, so the second-order and higher integrals are neglected here. With the similarity transformation, the canonical coordinates, $x(s)$ and $y(s)$, are expressed at the center of the quad, for $K_2 > 0$,

$$\begin{aligned} x(s) &= \cos\left(\sqrt{|K_2|}s\right)x_0 + \frac{1}{\sqrt{|K_2|}}\sin\left(\sqrt{|K_2|}s\right)p_{x,0} \\ y(s) &= \cosh\left(\sqrt{|K_2|}s\right)y_0 + \frac{1}{\sqrt{|K_2|}}\sinh\left(\sqrt{|K_2|}s\right)p_{y,0} \end{aligned}, \tag{5}$$

where the subscript "0" means the relative coordinates at $s = 0$, the center of the quad. Substituting Eq. (5) into Eq. (4) and performing the integral, the generator F_δ has the form

$$\begin{aligned} F_\delta = \frac{1}{4}K_2 l\delta\Bigg[& x_0^2\left(1 + \frac{\sin\phi}{\phi}\right) + p_{x,0}^2\frac{1}{|K_2|}\left(1 - \frac{\sin\phi}{\phi}\right) \\ & - y_0^2\left(1 + \frac{\sinh\phi}{\phi}\right) - p_{y,0}^2\frac{1}{|K_2|}\left(1 - \frac{\sinh\phi}{\phi}\right)\Bigg] \end{aligned} \tag{6}$$

where $\phi = \sqrt{|K_2|}l$,

in agreement with the result given in Ref. [6]. On the R.H.S. of Eq. (6), the square terms, including p_x and p_y, are the higher-order effect of a thick quad. They are smaller than other terms by a factor $|K_2|^{-1}\beta^{-2}$. In our case, it is at least equal to 10^{-2} and neglected here. We can also use the same procedures to calculate F_δ for the case of $K_2 < 0$. Expressing F_δ in terms of action angle coordinates and averaging the F_δ over the betatron phases, $\Phi_{x,y}$, we have

$$\langle F_\delta \rangle = \begin{cases} \frac{1}{8\pi}K_2 l\delta\left[\left(1 + \frac{\sin\phi}{\phi}\right)J_x\beta_x - \left(1 + \frac{\sinh\phi}{\phi}\right)J_y\beta_y\right] & (K_2 > 0) \\ \frac{1}{8\pi}K_2 l\delta\left[\left(1 + \frac{\sinh\phi}{\phi}\right)J_x\beta_x - \left(1 + \frac{\sin\phi}{\phi}\right)J_y\beta_y\right] & (K_2 < 0) \end{cases}, \tag{7}$$

where $\langle F \rangle_\delta$ denotes the perturbed Hamiltonian representing the effect of fringe fields interference. The betatron tune shifts can be solved according to

Table 1 The changes in tunes obtained via three methods

Tune shifts	Thin-lens quad model	Thick-lens quad model	MAD tracking
$\Delta\mu_x$	−0.0253	−0.0256	−0.026612
$\Delta\mu_y$	−0.0339	−0.0336	−0.033566

$$\Delta\mu_x = \frac{\Delta\Phi_x}{2\pi} = \frac{1}{2\pi}[\langle F_\delta\rangle, \Phi_x] = \frac{1}{2\pi}\frac{\partial\langle F_\delta\rangle}{\partial J_x}, \tag{8}$$

where the Φ_x is the horizontal phase advance and J_x is the action coordinate. According to Eq. (8), the derivatives of $\langle F\rangle_\delta$ with respect to $J_{x,y}$ yield the betatron tune shifts:

$$\begin{cases} \Delta\mu_x = -\dfrac{1}{8\pi}\displaystyle\sum_Q K_2 l\delta\left[\left(1+\dfrac{\sin\phi}{\phi}\right)\beta_x\right] \\ \Delta\mu_y = \dfrac{1}{8\pi}\displaystyle\sum_Q K_2 l\delta\left[\left(1+\dfrac{\sinh\phi}{\phi}\right)\beta_y\right] \end{cases} \quad (\text{for } K_2 > 0),$$
$$\begin{cases} \Delta\mu_x = -\dfrac{1}{8\pi}\displaystyle\sum_Q K_2 l\delta\left[\left(1+\dfrac{\sinh\phi}{\phi}\right)\beta_x\right] \\ \Delta\mu_y = \dfrac{1}{8\pi}\displaystyle\sum_Q K_2 l\delta\left[\left(1+\dfrac{\sin\phi}{\phi}\right)\beta_y\right] \end{cases} \quad (\text{for } K_2 < 0) \tag{9}$$

where the sums are over all quads interfered with the adjacent sexts. For the common strength quads, it is a good approximation to set

$$1+\frac{\sin\phi}{\phi} \approx 1+\frac{\sinh\phi}{\phi} \approx 2, \tag{10}$$

then the betatron tune shifts are simplified to the case of a thin-lens quad [7]

$$\begin{Bmatrix} \Delta\mu_x \\ \Delta\mu_y \end{Bmatrix} = \begin{Bmatrix} \pm \\ \mp \end{Bmatrix} \frac{1}{4\pi}\sum_Q \beta_{x,y}\Delta(K_2 l)\,, \tag{11}$$

where the upper sign for focusing quads is $K_2 > 0$ and the lower sign for defocusing quads is $K_2 < 0$.

3 Application to BEPC-II

In the BEPC-II arcs, the values of $\sin\phi/\phi$ are between 0.971 and 0.987 and the values of $\sinh\phi/\phi$ are between 1.012 and 1.028. This means that Eq. (9) furnishes a correction of about 3 % to the betatron tune shift calculated via Eq. (11). To verify this conclusion, we change the strengths of 36 quads in arcs by 0.57 %, evaluate the betatron tune shifts obtained from both Eqs. (9) and (11), and finally compare corresponding results with MAD, respectively. The tune shifts calculated via the three methods are listed in Table 1. It can be seen that the tune shifts obtained based on the thick quad model are more accurate than theories based on the thin quad. This verifies the conclusion. Usually, it is a reasonably good approximation to treat a practical quad as a thin-lens element if these quads are not sufficiently strong. But if this is not so, the corrections must be taken into account. For instance, we consider a typical quad with a gradient of 15 T/m and a length of 0.5 m. The correction factors $\sin\phi/\phi$ and $\sinh\phi/\phi$ arrive at about 10 %, and this cannot be neglected, obviously. In the BEPC-II ring ($\mu_x = 7.51$, $\mu_y = 5.56$), the average K for focusing Quads is a bit larger than the average K for defocusing Quads. That is why there is still a relatively larger difference between the thick-lens quad model and MAD tracking in the horizontal direction.

4 Conclusion

In this paper, an analytical expression of the tune shift due to quad error is obtained based on the thick-lens quad model and Lie algebra techniques, and the results verify that this expression will give a more accurate evaluation. A small correction needs to be considered when the thin-lens quad model is no longer applicable. The difference between the thick-lens quad model and MAD tracking will be studied in further works.

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