A general fan-beam reconstruction algorithm for free-form trajectories

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Abstract In this paper we develop a general exact fan-beam reconstruction algorithm for free-form trajectories not only closed but also unclosed, based on the fan-beam reconstruction formula recently developed by Noo *et al.*. A mathematical proof is then provided with the geometrical explanation of equi-spatial detectors. With this algorithm we can obtain exact region of interest (ROI) reconstruction if and only if every projecting line passing through the ROI intersects the free-form source trajectory, when the projections are not truncated. Furthermore, under the condition that the source-to-detectors distance changes slowly enough relative to the length itself, we obtain a very good approximate reconstruction algorithm, which is the same as the algorithm of the circular trajectory except that the source-to-detectors distance is a function of the rotation angle. Then the algorithms are tested using the Shepp-Logan phantom and the experiment shows that the algorithms can get perfect numerical results.

Key words Tomography, Fan-beam, Image reconstruction, Region of Interest (ROI)

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1 Introduction

Fan-beam reconstruction was firstly studied in the case of the closed circular-scanning trajectory,[1,2] and then extended to the case of closed noncircular scanning trajectories.[3-6] But all these fan-beam reconstruction algorithms need closed or half scanning trajectories. Recently, Noo *et al.* re-formulated a 2D FBP-type reconstruction algorithm of a region of interest (ROI) from the X-ray fan-beam projections.[7] It showed that a ROI could be exactly reconstructed only using super-short-scan projections. It is well known that images can be reconstructed from projections acquired just $\pi+2\gamma_{max}$ in fan-beam computed tomography (CT) using half-scan algorithm. By contrast with the half-scan method,[8] it is feasible that a ROI can be accurately reconstructed only provided all the projections from every line passing through the ROI, instead of the whole object support.

Subsequently, Kudo extended the super-short-scan algorithm to the cone-beam tomography.[9] But all the existing super-short-scan algorithms require a circular trajectory. In this paper, a general exact fan-beam reconstruction algorithm for free-form orbits is presented and proved. Under some easily satisfied conditions, we expand a very good approximate reconstruction algorithm which is derivative-free. It is clear that our algorithm possesses various potential benefits such as simplification and agility of data acquisition, good temporal resolution, reduced patient dose, lowered X-ray source power and reduction of computational requirements. Fig.1 shows some different free-form trajectories corresponding to different ROI that can be exactly reconstructed.

2 New algorithm for free-form trajectories

This section describes the FBP-style algorithm

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which is exact for fan-beam

ROI reconstruction with closed or unclosed free-form trajectories. The derivation is along the similar lines to Ref. [7]. In the fan-beam case, let $f(\vec{x})$ denote the 2-D object composed by $\vec{x}=(x_1,x_2)^T$. We assume that the X-ray source moves along the free-form trajectory $\vec{a}(\lambda)$ and $a(\lambda)$ denote the distance between the X-ray source and the origin; $\lambda\in\Lambda$. The source-to-detectors distance is a function of λ, denoted by $D(\lambda)$. We also assume the unit vector $\vec{e}_1=-\vec{a}(\lambda)/\|\vec{a}(\lambda)\|$ and $\vec{e}_2\perp\vec{e}_1$. Every projection is non-truncated and the detectors line is always orthogonal to the source-to-origin line (Fig.2). Finally, let $g(\lambda,u)$ denote the measured equi-spatial fan-beam projections. We also denote the fan-beam transform:

$$g(\lambda,u)=\int_0^{+\infty} f(\vec{a}(\lambda)+tu)\cdot \mathrm{d}u \qquad (1)$$

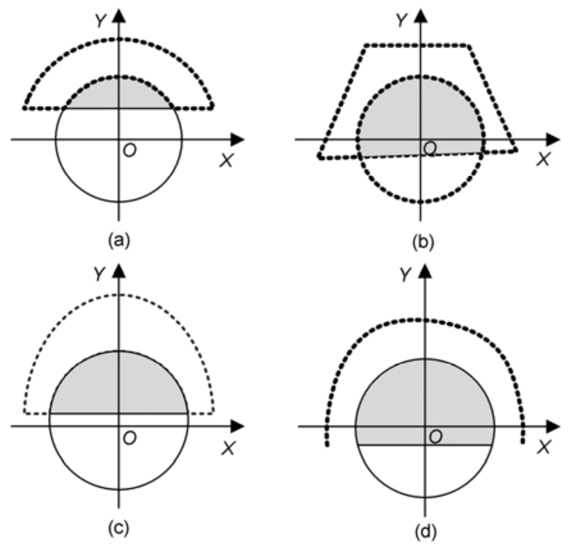

Fig.1 Some scanning loci (dotted lines) corresponding to different ROI (shadowed segments) which can be exactly reconstructed except (b). The ROI segments depend on the beginning and ending points of the scanning locus.

Then we can propose the following exact reconstruction algorithm for any free-form trajectories:

$$f(\vec{x})=\frac{1}{2\pi}\int_\Lambda \frac{1}{a(\lambda)+\vec{x}\cdot\vec{e}_1}\times [w(\lambda,\tilde{u})\cdot g_F(\lambda,\tilde{u})]_{\tilde{u}=\tilde{u}^*(\lambda,\vec{x})}\mathrm{d}\lambda \qquad (2)$$

with

$$g_F(\lambda,\tilde{u})=\int_{-u_m}^{u_m} h_H(\tilde{u}-u)\frac{D(\lambda)}{\sqrt{D^2(\lambda)+u^2}}\times \left(\frac{\partial}{\partial\lambda}+\frac{D'(\lambda)\cdot u+D^2(\lambda)+u^2}{D(\lambda)}\times\frac{\partial}{\partial u}\right)\times \qquad (3)$$

where $h_H(\bullet)$ in Eq. (3) represents the Hilbert transform kernel defined as in Refs. [7,10]:

$$h_H(u)=-\int_{-\infty}^{+\infty}\mathrm{i}sign(\sigma)\mathrm{e}^{\mathrm{i}2\pi\sigma u}\mathrm{d}\sigma=\frac{1}{\pi u} \qquad (4)$$

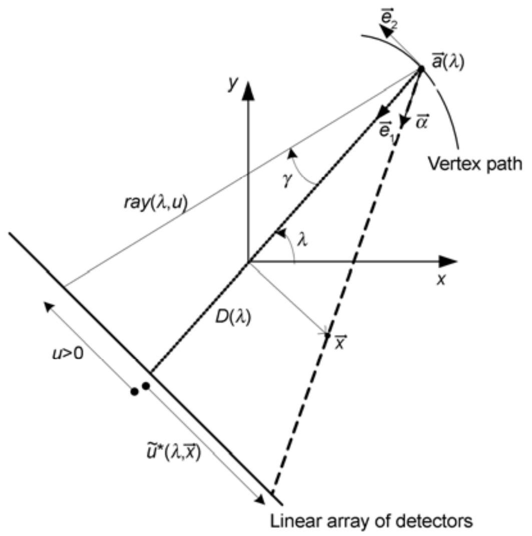

Fig.2 Fan-beam geometry with equally spaced collinear detectors.

In Eq.(2), $w(\lambda,\tilde{u})$ is a weight used to account for information redundancy in the dataset. We know that $(\lambda,\tilde{u})$ can identify an X-ray. And let $N(\lambda,\tilde{u})\in N$ be the number of intersections of the line $(\lambda,\tilde{u})$ with the vertex path. For lines through the ROI, $N(\lambda,\tilde{u})\geq 1$ and for any such fixed line, the intersections labeled $\vec{a}(\lambda_i), i=1,\ldots,N(\lambda,\tilde{u})$. For each line through the ROI, a weighting function $\hat{w}_{(\lambda,\tilde{u})}$ is defined as[7]

$$\sum_{i=1}^{N(\lambda,\tilde{u})}\hat{w}_{(\lambda,\tilde{u})}(\lambda_i)=1 \qquad (5)$$

Expressed in terms of weights on projection values $w(\lambda,\tilde{u}) = \hat{w}_{(\lambda,\tilde{u})}(\lambda)$.

For any free-form planar source trajectories, the ROI including the trajectories can be exactly reconstructed using Eq.(2), if the projections are non-truncated. In Eq.(3) there is a derivative of the source-to-detectors which is not easy to be achieved in applications.

In a general way the source-to-detectors distance changes slowly and $D'(\lambda)$ is a very small value in contrast to $D(\lambda)$. So we can get the conditions that:

1) $D'(\lambda)$ exists almost everywhere;

2) $D'(\lambda)/D(\lambda) \approx 0$

Actually, the second condition is easily satisfied in the practice, because in general $D(\lambda)$ is much greater than $D'(\lambda)$.

Then we can obtain a derivative-free algorithm, which is very approximate to Eqs. (2) and (3) as follows:

$$f(\vec{x}) = \frac{1}{2\pi}\int_{\Lambda} \frac{1}{a(\lambda)+\vec{x}\cdot\vec{e}_1} \times \left[w(\lambda,\tilde{u})\cdot g_F(\lambda,\tilde{u})\right]_{\tilde{u}=\tilde{u}^*(\lambda,\vec{x})} \mathrm{d}\lambda \quad (6)$$

$$g_F(\lambda,\tilde{u}) = \int_{-u_m}^{u_m} h_H(\tilde{u}-u)\frac{D(\lambda)}{\sqrt{D^2(\lambda)+u^2}} \times \left[\frac{\partial}{\partial\lambda} + \frac{D^2(\lambda)+u^2}{D(\lambda)} \times \frac{\partial}{\partial u}\right] \times g(\lambda,u)\mathrm{d}u \quad (7)$$

According to the formulas (37), (38) in Ref. [7], the difference between our derivative-free fan-beam algorithm and the Noo's just lies in the definition of *D*. *D* is a function of $\lambda \in \Lambda$ in the former, while a constant in the latter.

3 Mathematical proof

Our fan-beam reconstruction formulas (2) and (3) for any free-form trajectory are accurate under the condition that $D'(\lambda)$ always exists as $\lambda \in \Lambda$, that is, the source-moving curve is smooth everywhere.

Let us start with the following ROI fan-beam reconstruction formula (Eq. (27) of Ref. [7]):

$$f(\vec{x}) = \frac{1}{2\pi}\int_{\Lambda} \frac{1}{\|\vec{x}-\vec{a}(\lambda)\|} \times \left[w(\lambda,\vec{n}) \times g_F(\lambda,\vec{n})\right]_{\vec{n}=\vec{n}^*(\lambda,\vec{x})} \mathrm{d}\lambda \quad (8)$$

$$g_F(\lambda,n(\varphi)) = \int_{-\pi}^{\pi} h_H[\sin(\varphi-\gamma)]g'[\lambda,\alpha(\gamma)]\mathrm{d}\gamma \quad (9)$$

Under the fan-beam geometry with equally spaced collinear detectors (Fig.2.) we define

$$u = D(\lambda)\tan\gamma, \quad \tilde{u} = D(\lambda)\tan\varphi \quad (10)$$

Then from Eqs. (8), (9) we can obtain the new ROI fan-beam reconstruction algorithm for free-form trajectories

$$f(\vec{x}) = \frac{1}{2\pi}\int_{\Lambda} \frac{1}{\|\vec{x}-\vec{a}(\lambda)\|} \times \left[w(\lambda,\tilde{u}) \times g_F(\lambda,\tilde{u})\right]_{\tilde{u}=\tilde{u}^*(\lambda,\vec{x})} \mathrm{d}\lambda \quad (11)$$

$$g_F(\lambda,\tilde{u}) = \int_{\Lambda} \frac{\mathrm{d}\gamma}{\mathrm{d}u} h_H \times \left[\sin(\arctan\frac{\tilde{u}}{D(\lambda)} - \arctan\frac{u}{D(\lambda)})\right] \times \left(\frac{\partial}{\partial\lambda} + \frac{\partial}{\partial u} \times \frac{\partial u}{\partial\lambda}\right) g(\lambda,u)\mathrm{d}u \quad (12)$$

To prove Eq. (11) , it is only necessary to show that

$$g_F(\lambda,\tilde{u})|_{\tilde{u}=\tilde{u}^*(\lambda,\vec{x})} = g_F(\lambda,\vec{n})|_{\vec{n}=\vec{n}^*(\lambda,\vec{x})} \quad (13)$$

with g_F given by

$$g_F(\lambda,n(\varphi)) = \int_{-\pi}^{\pi} h_H\left[\sin(\varphi-\gamma)\right] g'\left[\lambda,\alpha(\gamma)\right]\mathrm{d}\gamma$$

Under the fan-beam geometry with equally spaced collinear detectors, we define

$$u = D(\lambda)\tan\gamma \quad (14)$$

$$\tilde{u} = D(\lambda)\tan\varphi \quad (15)$$

$$\tilde{u}^* = D(\lambda)\tan\varphi^* \quad (16)$$

Suggested in Eq. (14),

$$\gamma = \arctan\frac{u}{D(\lambda)} \quad (17)$$

In the given Eq. (9),

$$g' = \frac{\partial g}{\partial \lambda} + \frac{\partial g}{\partial \gamma} \times \frac{\partial \gamma}{\partial u} \times \frac{\partial u}{\partial \lambda} = \left(\frac{\partial}{\partial \lambda} + \frac{\partial}{\partial u} \times \frac{\partial u}{\partial \lambda}\right) g(\lambda, u) \tag{18}$$

Hence,

$$g'(\lambda, \alpha(\gamma)) = \left(\frac{\partial}{\partial \lambda} + \frac{\partial}{\partial u} \times \frac{\partial u}{\partial \lambda}\right) g(\lambda, u) \tag{19}$$

And from Eq. (9),

$$g_F(\lambda, n(\varphi)) = \int_{-\pi}^{\pi} h_H\left[\sin(\varphi - \gamma)\right] \times \left(\frac{\partial}{\partial \lambda} + \frac{\partial}{\partial u} \times \frac{\partial u}{\partial \lambda}\right) g(\lambda, u) \mathrm{d}\gamma = g_F(\lambda, u) \tag{20}$$

Using Eqs. (14) and (15) for Eq. (20), we can also get Eq. (12).

Now, it only remains to be shown that $\tilde{u}^*(\lambda, \vec{x})$ is such that $\vec{n}[\tilde{u}^*(\lambda, \vec{x})] = \vec{n}^*(\lambda, \vec{x})$. According to Eq. (58) and (65) in Ref. [7],

$$\vec{n}^*(\lambda, \vec{x}) = \vec{n}[\varphi^*(\lambda, \vec{x})] \tag{21}$$

Substitute (16) into (21), we conclude that

$$\vec{n}^*(\lambda, \vec{x}) = \vec{n}[\tilde{u}^*(\lambda, x)] \tag{22}$$

So Eqs. (11) and (13) are proved.

From Eq. (10) we can get

$$\frac{\mathrm{d}\gamma}{\mathrm{d}u} = \frac{D(\lambda)}{D^2(\lambda) + u^2} \tag{23}$$

$$\frac{\partial}{\partial \lambda} + \frac{\partial}{\partial u} \times \frac{\partial u}{\partial \lambda} = \frac{\partial}{\partial \lambda} + \frac{\partial}{\partial u} \times \frac{\partial u}{\partial \gamma} \times \frac{\partial \gamma}{\partial \lambda} = \frac{\partial}{\partial \lambda} + \frac{D'(\lambda) \cdot u + D^2(\lambda) + u^2}{D(\lambda)} \times \frac{\partial}{\partial u} \tag{24}$$

In above proof we have used Eq. (61) of Ref. [7].

Based on the relationship (10) the Hilbert transform kernel can be converted into

$$h_H\left[\sin(\arctan\frac{\tilde{u}}{D(\lambda)} - \arctan\frac{u}{D(\lambda)})\right] = h_H\left[\frac{D(\lambda) \times (\tilde{u} - u)}{\sqrt{\tilde{u}^2 + D^2(\lambda)} \times \sqrt{u^2 + D^2(\lambda)}}\right] \tag{25}$$

Based on the $h_H(\bullet)$'s definition Eq. (4), we define $\sigma' = \sigma \frac{D(\lambda)}{\sqrt{\left[\tilde{u}^2 + D^2(\lambda)\right]\left[u^2 + D^2(\lambda)\right]}}$. Hence, Eq. (15) can become

$$h_H\left[\frac{D(\lambda) \times (\tilde{u} - u)}{\sqrt{\left[\tilde{u}^2 + D^2(\lambda)\right]\left[u^2 + D^2(\lambda)\right]}}\right] = -\int_{-\infty}^{+\infty} \mathrm{i}sign(\sigma) \mathrm{e}^{\mathrm{i}2\pi\sigma \frac{D(\lambda)\times(\tilde{u}-u)}{\sqrt{\left[\tilde{u}^2+D^2(\lambda)\right]\left[u^2+D^2(\lambda)\right]}}} \mathrm{d}\sigma$$
$$= -\frac{\sqrt{\left[\tilde{u}^2 + D^2(\lambda)\right]\left[u^2 + D^2(\lambda)\right]}}{D(\lambda)} \times \int_{-\infty}^{+\infty} \mathrm{i}sign(\sigma') \mathrm{e}^{\mathrm{i}2\pi\sigma'(\tilde{u}-u)} \mathrm{d}\sigma' = \frac{\sqrt{\left[\tilde{u}^2 + D^2(\lambda)\right]\left[u^2 + D^2(\lambda)\right]}}{D(\lambda)} \times h_H(\tilde{u} - u) \tag{26}$$

Then making use of Eqs. (13) ~ (16) in Eq. (12) we obtain

$$g_F(\lambda, \tilde{u}) = \int_{\Lambda} \frac{D(\lambda)}{\sqrt{D^2(\lambda) + u^2}} \times \frac{\sqrt{\left[D^2(\lambda) + \tilde{u}^2\right]\left[D^2(\lambda) + u^2\right]}}{D(\lambda)} \times h_H(\tilde{u} - u) \times \left[\frac{\partial}{\partial \lambda} + \frac{D'(\lambda) \times u + D^2(\lambda) + u^2}{D(\lambda)} \times \frac{\partial}{\partial u}\right] \times g(\lambda, u) \mathrm{d}u \tag{27}$$

We take $\frac{\sqrt{\left[\tilde{u}^2 + D^2(\lambda)\right]}}{D(\lambda)}$ out of the integral about $\mathrm{d}u$ and define

$$g_F(\lambda, \tilde{u}) = \int_{\Lambda} \frac{D(\lambda)}{\sqrt{D^2(\lambda) + u^2}} \times h_H(\tilde{u} - u) \times \left[\frac{\partial}{\partial \lambda} + \frac{D'(\lambda) \times u + D^2(\lambda) + u^2}{D(\lambda)} \times \frac{\partial}{\partial u}\right] \times g(\lambda, u) \mathrm{d}u \tag{28}$$

Using Eq. (18) for Eq. (11) we get

$$f(\vec{x}) = \frac{1}{2\pi} \int_{\Lambda} \frac{1}{\left\|\vec{x} - \vec{a}(\lambda)\right\|} \times \frac{\sqrt{(\tilde{u}^2 + D^2(\lambda))}}{D(\lambda)} \times \left[w(\lambda, \tilde{u}) g_F(\lambda, \tilde{u})\right]_{\tilde{u} = \tilde{u}^*(\lambda, \vec{x})} \mathrm{d}\lambda \tag{29}$$

Under the fan-beam geometry with equi-spatial collinear detectors (Fig. 2.) we can get

$$\frac{1}{\left\|\vec{x}-\vec{a}(\lambda)\right\|}\times\frac{\sqrt{\left[\tilde{u}^2+D^2(\lambda)\right]}}{D(\lambda)}=\frac{1}{a(\lambda)+\vec{x}\cdot\vec{e}_1} \quad (30)$$

Thus, we have proved the exact reconstruction formulas (6) and (7) for any free-form trajectories.

4 Simulation results

Our fan-beam ROI reconstruction algorithm for any free-form trajectories is numerically evaluated using the 2-D Shepp-Logan phantom. There are ten ellipses in the phantom. The simulation parameters are summarized in Table 1. The new ROI reconstruction algorithm is simulated with an elliptical trajectory, scanning for full-scan and super-short-scan datasets, that is, 2π and π. The detectors line is always orthogonal to the source-to-origin line.

Table 1 Fan-beam imaging parameters used in numerical simulation

Object radius R_0	2 cm
Semi-major axis of elliptical locus R_a	6 cm
Semi-major axis of elliptical locus R_b	5 cm
Origin-to-detectors distance	1 cm
Detector array size	256
Fan-beam angle γ_m	23°
Projection per 2π	180
Reconstruction matrix	256×256
Detector width (Δu)	0.00956 cm

The derivatives with respect to γ and u are implemented using the 2-point formula. The weight function $w(\lambda,\tilde{u})$ is defined as follows:

$$w(\lambda,\tilde{u})=\frac{c(\lambda)}{c(\lambda)+c\left[\lambda+\pi-2\arctan(\tilde{u}/D(\lambda))\right]} \quad (31)$$

with $c(\lambda)$ defined as Ref. [11]:

$$c(\lambda)=\begin{cases}\cos^2\dfrac{\pi(\lambda-\lambda_s-d)}{2d} & (\lambda_s<\lambda<\lambda_s+d)\\ 1 & (\lambda_s+d<\lambda<\lambda_e-d)\\ \cos^2\dfrac{\pi(\lambda-\lambda_e+d)}{2d} & (\lambda_e-d<\lambda<\lambda_e)\\ 0 & (\text{others})\end{cases} \quad (32)$$

where λ_s and λ_e are respectively the starting and ending points of the scanning locus segment, and d is an angular interval over which $c(\lambda)$ smooth drops from 1 to 0. Here $d=23^\circ$ is used.

In Fig.3 some numerical results are shown.

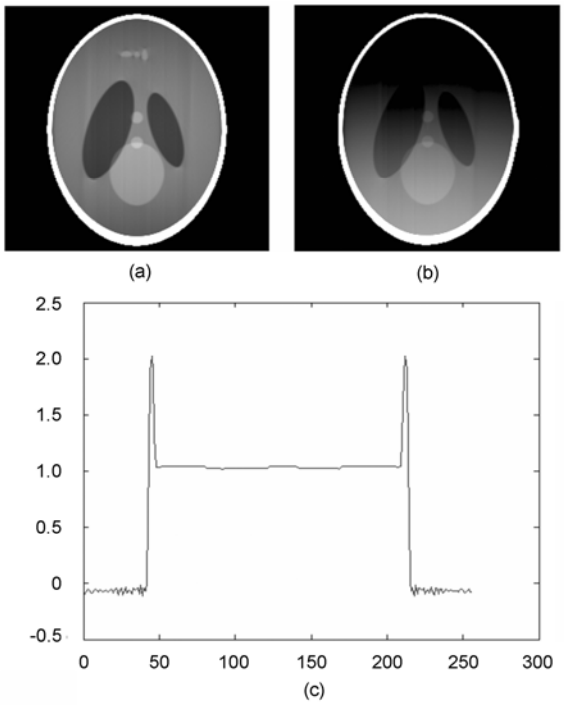

Fig.3 Fan-beam reconstruction of elliptical locus. (a) Reconstructed Shepp-Logan phantom image using projections of 2π. (b) Reconstructed Shepp-Logan phantom image using projections of π. (c) One line of reconstructed Shepp-Logan phantom image using projection of π.

5 Discussions and conclusion

In the view of this paper, we have proposed fan-beam FBP-type reconstruction algorithms including an exact one and a very approximate one for any free-form trajectories. With our proof we can get accurate ROI images by arbitrary scanning locus. However, we would like to emphasize that our algorithm is exact only with a smooth scanning locus. And for easy to practical application we simplify it to a derivative-free algorithm which is good approximate under some easily satisfied conditions and very similar to Noo's algorithm. Because the algorithms are FBP-type they can realize fast image reconstruction and are easily expanded.

As the above description the algorithm can only be used when the source locus is in a plane. But inspired by PI-method,[12,13] especially the algorithms for cone-beam spiral CT recently proposed by Katsevich,[14-16] we suggest the algorithm be expended to cone-beam tomography with any free-form locus in 3-D spaces.

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