The electromagnetic Casimir effect of spherical cavity

HAN Jia-Guang, YU Li-Ping, WANG Zhen-Xia, ZHU Zhi-Yuan
(Shanghai Institute of Nuclear Research, the Chinese Academy of Sciences, Shanghai 201800)

Abstract The Casimir effect results from the zero-point energy of vacuum. A spherical cavity can be divided into three regions, and we make an analysis of every region and then give a formal solution of Casimir energy. The zeta-function regularization is also used to dispel the divergence of the summation. At the end, we can see the Casimir effect of a single sphere is included in our results.

Keywords Casimir effect, Zeta-function regularization, Spherical cavity, Casimir energy
CLC numbers O442, O413.2

1 Introduction

The interest about Casimir effect has been steadily grown due to its application in different domains of physics since Casimir published his famous paper.[1] As we know, the zero-point energies from vacuum is $\frac{1}{2}\hbar\sum_n \omega_n$, which gives rise to the Casimir effect, where the index n labels the quantum numbers of field modes.[2,3] It is easy to see that the summation is divergent, and in order to remove the divergence we usually use the zeta-function regularization,[4,5] which is denoted by

$$E(s)=\sum_n {\omega_n}^{-s}$$

and the limit of regularization is $s\to -1$. Then we can get the Casimir energy E_c[5] defined by:

$$E_{\mathrm{c}}=\lim_{s\to -1}\mathrm{PP}[\frac{1}{2}\hbar E(s)]$$

where PP denotes principal parts.

Boyer[4] has made some research on Casimir effect of a conducting spherical shell but he didn't use the zeta-function regularization. In order to smooth the divergent series, he introduced a spherical cavity model which consists of a large conducting sphere of radius R enclosing the quantization 'universe'. As a result, he got the zero-energy of a conducting spherical shell but not a spherical cavity. Our interest is to discuss directly the Casimir effect of a spherical cavity using the zeta-function regularization method. Recently Jiang and his collaborators[3] calculated the Casimir energy of a conducting cylindrical cavity with zeta-function regularization. In this paper, we report on a neutral and perfectly conducting spherical cavity's Casimir effect.

2 Calculation

2.1 Region I

We consider Region **I** (see Fig.1) firstly and the Maxwell equations are:

$$\nabla\cdot\vec{E}=0,\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{B}=0,\ \nabla\times\vec{B}=\frac{\partial\vec{E}}{\partial t}$$

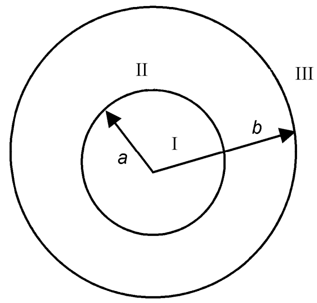

Fig.1 Three regions of the spherical cavity.

And for some certain frequency ω, from Maxwell equations we can get Helmhotz equations:

Supported by the Major Project of Knowledge Innovation Project of the Chinese Acedamy of Sciences (No.KJCX2-SW-N02)
Received date: 2002-09-19

$$\begin{cases}\nabla\cdot\vec{E}+k^2\vec{E}=0\\ \nabla\cdot\vec{E}=0\end{cases}\qquad\begin{cases}\nabla\cdot\vec{B}+k^2\vec{B}=0\\ \nabla\cdot\vec{B}=0\end{cases}$$

if we suppose the velocity of light c = 1 and then $k=\omega$. We will look for solutions in the spherical coordinates of the form:

$$E_i, B_i = f(r)g(\theta)q(\phi)$$

and then we have:

$$\begin{cases}[\dfrac{\mathrm{d}}{\mathrm{d}r}(r^2\dfrac{\mathrm{d}}{\mathrm{d}r})+(kr)^2-l(l+1)]f(r)=0\\ [\dfrac{1}{\sin\theta}\dfrac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta\dfrac{\mathrm{d}}{\mathrm{d}\theta})+l(l+1)-\dfrac{m^2}{\sin^2\theta}]g(\theta)=0\\ [\dfrac{\mathrm{d}^2}{\mathrm{d}\phi^2}+m^2]q(\phi)=0\end{cases}$$

The solution has radical parts $f(r)\propto r^{-\frac{1}{2}}J_\nu(\omega r)$, and the possible frequencies ω are determined by boundary conditions:[2, 3, 5]

$$r^{-\frac{1}{2}}J_\nu(\omega r)\big|_{r=a}=0$$, Dirichlet (D) condition, for Region **I**, TE-mode

$$\frac{\mathrm{d}}{\mathrm{d}r}(r^{\frac{1}{2}}J_\nu(\omega r))\big|_{r=a}=0$$, Robin (R) condition, for Region **I**, TM-mode

where $J_\nu(\omega r)$ is Bessel function,[6] $\nu=\frac{1}{2}+l$, and l is integral number. If we defineand $j_{\nu,n}$ and $k_{\nu,n}$ are the n-th nonzero solutions of the above equations separately under D and R condition, we can get:

$$E^{\mathrm{I,D}}(s)=\sum_n\sum_l 2a^s\cdot(l+\frac{1}{2})\cdot j_{\nu,n}{}^{-s}$$

$$E^{\mathrm{I,R}}(s)=\sum_n\sum_l 2a^s\cdot(l+\frac{1}{2})\cdot k_{\nu,n}{}^{-s}$$

2.2 Region III

The same calculation is also useful for Region III, but the radial parts now are $\propto r^{-\frac{1}{2}}H^{(1)}{}_\nu(\omega r)$, where $H^{(1)}{}_\nu(\omega r)$ is Hankel function.[6] After taking the same procedure as Region **I** and defining $j'_{\nu,n}$ and $k'_{\nu,n}$ as the n-th nonzero solutions of $H^{(1)}{}_\nu(\omega r)$, we have:

$$E^{\mathrm{III,D}}(s)=\sum_n\sum_l 2b^s\cdot(l+\frac{1}{2})\cdot j'_{\nu,n}{}^{-s}$$

$$E^{\mathrm{III,R}}(s)=\sum_n\sum_l 2b^s\cdot(l+\frac{1}{2})\cdot k'_{\nu,n}{}^{-s}$$

2.3 Region II

Finally we consider the Region **II**. Because its solution can not be expressed simply by $J_\nu(\omega r)$ or $H^{(1)}{}_\nu(\omega r)$, the solution is denoted by $Ah_l^{(1)}(\omega r)+Bh_l^{(2)}(\omega r)$, where $h_l^{(1)}(\omega r)$ and $h_l^{(2)}(\omega r)$ are spherical Hankel function;[5] A and B are coefficients. We know there are equations as follows:[3, 5,6]

$$\begin{cases}Ah_l^{(1)}(\omega a)+Bh_l^{(2)}(\omega a)=0\\ Ah_l^{(1)}(\omega b)+Bh_l^{(2)}(\omega b)=0\end{cases}\qquad \text{D condition}$$

$$\begin{cases}Ah_l^{(1)\prime}(\omega a)+Bh_l^{(2)\prime}(\omega a)=0\\ Ah_l^{(1)\prime}(\omega b)+Bh_l^{(2)\prime}(\omega b)=0\end{cases}\qquad \text{R condition}$$

If these equations have solutions, the following formulas are necessary:

$$f^{\mathrm{II,D}}(\omega)=h_l^{(1)}(\omega a)h_l^{(2)}(\omega b)-h_l^{(1)}(\omega b)h_l^{(2)}(\omega a)=0$$

$$f^{\mathrm{II,R}}(\omega)=h_l^{(1)\prime}(\omega a)h_l^{(2)\prime}(\omega b)-h_l^{(1)\prime}(\omega b)h_l^{(2)\prime}(\omega a)=0$$

We define that $j''_{\nu,n}$ and $k''_{\nu,n}$ are the solution of $f^{\mathrm{II,D}}(\omega)$ =0 and $f^{\mathrm{II,R}}(\omega)$ =0 respectively, and then:

$$E^{\mathrm{II,D}}(s)=\sum_n\sum_l 2\cdot(l+\frac{1}{2})\cdot j''_{\nu,n}{}^{-s}$$

$$E^{\mathrm{II,R}}(s)=\sum_n\sum_l 2\cdot(l+\frac{1}{2})\cdot k''_{\nu,n}{}^{-s}$$

3 Results

After zeta-function regularization, we can get the results of Region **I** and Region **III**, and next we make an analysis of Region **II**.[4, 7-10]

$$E^{\mathrm{I,D}} = \frac{1}{a}[\frac{2}{315\pi}(\frac{1}{s+1}+\ln a)+0.27069+O(s+1)]$$

$$E^{\mathrm{I,R}} = \frac{1}{a}[\frac{2}{45\pi}(\frac{1}{s+1}+\ln a)-0.10285+O(s+1)]$$

$$E^{\mathrm{III,D}} = \frac{1}{b}[-\frac{2}{315\pi}(\frac{1}{s+1}+\ln b)-0.00326+O(s+1)]$$

$$E^{\mathrm{III,R}} = \frac{1}{b}[-\frac{2}{45\pi}(\frac{1}{s+1}+\ln b)-0.07223+O(s+1)]$$

$$E^{\mathrm{I}}(s) = E^{\mathrm{I,D}}(s)+E^{\mathrm{I,R}}(s) = \frac{1}{a}[\frac{16}{315\pi}(\frac{1}{s+1}+\ln a)+0.16784+O(s+1)]$$

$$E^{\mathrm{III}}(s) = E^{\mathrm{III,D}}(s)+E^{\mathrm{III,R}}(s) = \frac{1}{b}[-\frac{16}{315\pi}(\frac{1}{s+1}+\ln b)+0.07549+O(s+1)]$$

If radius of Region **II** $b\to\infty$, Region **II** will become Region **III**; but if $b\to 0$, we believe $E^{\mathrm{II}}(s)=E^{\mathrm{II,D}}(s)+E^{\mathrm{II,R}}(s) \to 0$ and now the spherical cavity also becomes the single sphere. So we have $0 \le E^{\mathrm{II}}(s) \le E^{\mathrm{III}}(s)$. In the end we can consider an e.m field in the whole space with the spherical cavity acting as a neutral and perfectly conducting boundary:

$$E_{\mathrm{c}}^{\mathrm{e\cdot m}} = \lim_{s\to -1}\mathrm{PP}[\frac{1}{2}\hbar(E^{\mathrm{I}}(s)+E^{\mathrm{II}}(s)+E^{\mathrm{III}}(s)]$$

4 Conclusion

The Casimir effect has become the subject of many physical fields especially with the development of submicroscopic physics. In this paper we introduce the Casimir effect of the spherical cavity and give a formal solution of Casimir energy. We would like to emphasize that there are still a lot of work to do and the Casimir effect is also exhibiting greater and greater importance on the development of modern physics.

References

1 Casimir H B G. Phys, 1956, **19**: 846

2 Bordag M, Mohideen U, Mostepanenko V M. Phys Rep, 2001, **353**: 1-205

3 Jiang W Z, Fu D J, Wang Z X *et al*. Acta Phys Sin (in Chinese), 2003, **52** (to be published)

4 Boyer T H. Phys Rev, 1968, **174**: 1764-1777

5 Leseduarte S, Romeo A. Hep-th, 9605022, 1996, **3**

6 Wang Z X, Guo D R. Introduction of mathematical functions (in Chinese), Beijing: Science Press, 1979

7 Elizalde E, Leseduarte S, Romeo A. J Phys A, 1994, **26**: 2409

8 Blau S K, Visser M, Wipf A. Nucl Phys B, 1988, **310**: 163

9 Leseduarte S, Romeo A. J Phys A , 1994, **27**: 2483

10 Gosdzinsky D, Romeo A. Phys Let B, 1998, **441**: 265