SOME USEFUL FORMULAE IN SU(N) ALGEBRA*

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ABSTRACT

Some useful relations in SU(N) algebra are given in this paper. These relations may be useful in deriving the many-body interaction in QCD.
Keywords SU(N) algebra, Many body interaction, Quantum chromodynamics, Hermitian operators, Many-dimensional calculations

1 INTRODUCTION

In particle physics, SU(N) algebra is an useful tool. For example, SU(2) is the basis of discussing the angular momentum , isospin and the weak-electromagnetic unification; SU(3) is the basis of discussing QCD and the flavour group of light quark; SU(4) is the basis of Dirac algebra; $\cdots$, and so on. In this paper, we follow the way introduced in Ref.[1] to give some useful relations in SU(N) algebra. Consider a set of N^2-1 traceless Hermitian operator $\lambda_k (k=1,2,...,N^2-1)$ defined a N-dimensional space.

$$\lambda_k^\dagger = \lambda_k \tag{1}$$

$$Tr(\lambda_k) = 0 \tag{2}$$

$$\lambda_k \lambda_l = \alpha \delta_{kl} \mathrm{I} + \beta_{klm} \lambda_m \tag{3}$$

$$\beta_{klm} = d_{klm} + i f_{klm} \tag{4}$$

$$[\lambda_k, \lambda_l] = \lambda_k \lambda_l - \lambda_l \lambda_k = 2 i f_{klm} \lambda_m \tag{5}$$

$$\{\lambda_k, \lambda_l\} = \lambda_k \lambda_l + \lambda_l \lambda_k = 2\alpha \delta_{kl} + 2 d_{klm} \lambda_m \tag{6}$$

Consider a set of column vectors; $\zeta_1, \zeta_2, \cdots$; and a set of row vectors; $\eta_1, \eta_2, \cdots$, $N \times N$ matrix $\zeta_1 \eta_1$, $\zeta_1 \eta_1 \zeta_2 \eta_2$, and $\zeta_1 \eta_1 \zeta_2 \eta_2 \zeta_3 \eta_3$ can be written as:

$$\zeta_1 \eta_1 = \frac{1}{N}(\eta_1 \zeta_1) + \frac{1}{N\alpha}(\eta_1 \lambda_k \zeta_1) \lambda_k \tag{7}$$

$$\begin{aligned} \zeta_1 \eta_1 \zeta_2 \eta_2 &= \frac{1}{N^2}(\eta_1 \zeta_1)(\eta_2 \zeta_2) + \frac{1}{N^2 \alpha}(\eta_1 \zeta_1)(\eta_2 \lambda_l \zeta_2) \lambda_l \\ &+ \frac{1}{N^2 \alpha}(\eta_1 \lambda_k \zeta_1)(\eta_2 \zeta_2) \lambda_k + \frac{1}{N^2 \alpha^2}(\eta_1 \lambda_k \zeta_1)(\eta_2 \lambda_l \zeta_1) \lambda_k \lambda_l \end{aligned} \tag{8}$$

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$$
\begin{aligned}
&\zeta_1\eta_1\zeta_2\eta_2\zeta_3\eta_3 \\
= &\frac{1}{N^3}(\eta_1\zeta_1)(\eta_2\zeta_2)(\eta_3\zeta_3) + \frac{1}{N^3\alpha}(\eta_1\zeta_1)(\eta_2\lambda_l\zeta_2)(\eta_3\zeta_3)\lambda_l \\
+ &\frac{1}{N^3\alpha}(\eta_1\lambda_k\zeta_1)(\eta_2\zeta_2)(\eta_3\zeta_3)\lambda_k + \frac{1}{N^3\alpha^2}(\eta_1\lambda_k\zeta_1)(\eta_2\lambda_l\zeta_2)(\eta_3\zeta_3)\lambda_k\lambda_l \\
+ &\frac{1}{N^3\alpha}(\eta_1\zeta_1)(\eta_2\zeta_2)(\eta_3\lambda_m\zeta_3)\lambda_m + \frac{1}{N^3\alpha^2}(\eta_1\zeta_1)(\eta_2\lambda_l\zeta_2)(\eta_3\lambda_m\zeta_3)\lambda_l\lambda_m \\
+ &\frac{1}{N^3\alpha^2}(\eta_1\lambda_k\zeta_1)(\eta_2\zeta_2)(\eta_3\lambda_m\zeta_3)\lambda_k\lambda_m + \frac{1}{N^3\alpha^3}(\eta_1\lambda_k\zeta_1)(\eta_2\lambda_l\zeta_2)(\eta_3\lambda_m\zeta_3)\lambda_k\lambda_l\lambda_m \quad (9)
\end{aligned}
$$

All these relations can be found in Ref.[1] except Eq.(9), which can be derived by the same technique used in Ref.[1] to obtain the relation . A summation conversion of repeated indices is assumed here.

2 DERIVATION

In order to give the useful formulae, we also need the following relations(see Ref.[1]):

$$(\lambda_k)_{a_1b_1}(\lambda_k)_{a_2b_2} = \alpha(N\delta_{a_1b_2}\delta_{a_2b_1} - \delta_{a_1b_1}\delta_{a_2b_2}) \quad (10)$$

$$
\begin{aligned}
&\beta_{klm}(\lambda_k)_{a_1b_1}(\lambda_l)_{a_2b_2}(\lambda_m)_{a_3b_3} \\
= &\alpha^2[N^2\delta_{a_1b_2}\delta_{a_2b_3}\delta_{a_3b_1} - N(\delta_{a_1b_1}\delta_{a_2b_3}\delta_{a_3b_2} + \delta_{a_1b_3}\delta_{a_2b_2}\delta_{a_3b_1} \\
&+\delta_{a_1b_2}\delta_{a_2b_1}\delta_{a_3b_3}) + 2\delta_{a_1b_1}\delta_{a_2b_2}\delta_{a_3b_3}] \quad (11)
\end{aligned}
$$

Consider further the following $N\times N$ matrix;

$$
\begin{aligned}
&(\eta_4\zeta_1)(\eta_1\zeta_2)(\eta_2\zeta_3)(\eta_3\zeta_4) \\
= &\frac{1}{N^3}(\eta_1\zeta_1)(\eta_2\zeta_2)(\eta_3\zeta_3)(\eta_4\zeta_4) + \frac{1}{N^3\alpha}(\eta_1\zeta_1)(\eta_2\lambda_l\zeta_2)(\eta_3\zeta_3)(\eta_4\lambda_l\zeta_4) \\
+ &\frac{1}{N^3\alpha}(\eta_1\lambda_k\zeta_1)(\eta_2\zeta_2)(\eta_3\zeta_3)(\eta_4\lambda_k\zeta_4) + \frac{1}{N^3\alpha^2}(\eta_1\lambda_k\zeta_1)(\eta_2\lambda_l\zeta_2)(\eta_3\zeta_3)(\eta_4\lambda_k\lambda_l\zeta_4) \\
+ &\frac{1}{N^3\alpha}(\eta_1\zeta_1)(\eta_2\zeta_2)(\eta_3\lambda_m\zeta_3)(\eta_4\lambda_m\zeta_4) + \frac{1}{N^3\alpha^2}(\eta_1\zeta_1)(\eta_2\lambda_l\zeta_2)(\eta_3\lambda_m\zeta_3)(\eta_4\lambda_l\lambda_m\zeta_4) \\
+ &\frac{1}{N^3\alpha^2}(\eta_1\lambda_k\zeta_1)(\eta_2\zeta_2)(\eta_3\lambda_m\zeta_3)(\eta_4\lambda_k\lambda_m\zeta_4) \\
+ &\frac{1}{N^3\alpha^3}(\eta_1\lambda_k\zeta_1)(\eta_2\lambda_l\zeta_2)(\eta_3\lambda_m\zeta_3)(\eta_4\lambda_k\lambda_l\lambda_m\zeta_4) \quad (12)
\end{aligned}
$$

where

$$
\begin{aligned}
(\eta_4\lambda_k\lambda_l\lambda_m\zeta_4) &= \eta_4(\alpha\delta_{kl}I + \beta_{kln}\lambda_n)\lambda_m\eta_4 \\
&= \alpha\delta_{kl}\eta_4\lambda_m\zeta_4 + \beta_{kln}\eta_4\lambda_n\lambda_m\zeta_4 \\
&= \alpha\delta_{kl}\eta_4\lambda_m\zeta_4 + \beta_{kln}\eta_4(\alpha\delta_{nm} + \beta_{nmo}\lambda_o)\zeta_4 \\
&= \alpha\delta_{kl}(\eta_4\lambda_m\zeta_4) + \alpha\delta_{nm}\beta_{kln}(\eta_4\zeta_4) + \beta_{kln}\beta_{nmo}(\eta_4\lambda_o\zeta_4) \quad (13)
\end{aligned}
$$

Substituting Eq.(11) into Eq.(12) and Eq.(10) into Eq.(11), after a tedious calculation, one has seen:

$$\begin{aligned}
&\beta_{kln}\beta_{nmo}(\lambda_k)_{a_1b_1}(\lambda_l)_{a_2b_2}(\lambda_m)_{a_3b_3}(\lambda_o)_{a_4b_4}\\
=\ &\alpha^3(N^3\delta_{a_1b_2}\delta_{a_2b_3}\delta_{a_3b_4}\delta_{a_4b_1} - N^2\delta_{a_1b_2}\delta_{a_2b_4}\delta_{a_3b_3}\delta_{a_4b_1} - N^2\delta_{a_1b_3}\delta_{a_2b_2}\delta_{a_3b_4}\delta_{a_4b_1}\\
&-N^2\delta_{a_1b_1}\delta_{a_2b_3}\delta_{a_3b_4}\delta_{a_4b_2} - N^2\delta_{a_1b_2}\delta_{a_2b_1}\delta_{a_3b_4}\delta_{a_4b_3} - N^2\delta_{a_1b_2}\delta_{a_2b_3}\delta_{a_3b_1}\delta_{a_4b_4}\\
&+2N\delta_{a_1b_1}\delta_{a_2b_2}\delta_{a_3b_4}\delta_{a_4b_3} + 2N\delta_{a_1b_2}\delta_{a_2b_1}\delta_{a_3b_3}\delta_{a_4b_4} + N\delta_{a_1b_4}\delta_{a_2b_2}\delta_{a_3b_3}\delta_{a_4b_1}\\
&+N\delta_{a_1b_1}\delta_{a_2b_4}\delta_{a_3b_3}\delta_{a_4b_2} + N\delta_{a_1b_1}\delta_{a_2b_3}\delta_{a_3b_2}\delta_{a_4b_4} + N\delta_{a_1b_3}\delta_{a_2b_2}\delta_{a_3b_1}\delta_{a_4b_4}\\
&-4\delta_{a_1b_1}\delta_{a_2b_2}\delta_{a_3b_3}\delta_{a_4b_4}) \qquad (14)
\end{aligned}$$

Setting $b_2 = a_4$ and sum over a_4, one has

$$\begin{aligned}
&\beta_{kln}\beta_{nmo}(\lambda_k)_{a_1b_1}(\lambda_l)_{a_2b_2}(\lambda_o\lambda_m)_{a_2b_4}\\
=\ &\alpha^3[-2N^2\delta_{a_1b_3}\delta_{a_3b_4}\delta_{a_2b_1} - N^2\delta_{a_1b_4}\delta_{a_2b}\delta_{a_3b_1} + 3N^2\delta_{a_1b_1}\delta_{a_2b_3}\delta_{a_3b_4}\\
&+3N\delta_{a_1b_4}\delta_{a_2b_1}\delta_{a_3b_3} + N\delta_{a_1b_3}\delta_{a_3b_1}\delta_{a_2b_4} - 4\delta_{a_1b_1}\delta_{a_3b_3}\delta_{a_2b_4}] \qquad (15)
\end{aligned}$$

Making use of Eq.(3) one has

$$\begin{aligned}
&\beta_{kln}\beta_{nmo}\beta_{lmp}(\lambda_k)_{a_1b_1}(\lambda_p)_{a_2b_2}(\lambda_o)_{a_4b_4}\\
=\ &\alpha^3[-2N^2\delta_{a_1b_3}\delta_{a_3b_4}\delta_{a_2b_1} - N^2\delta_{a_1b_4}\delta_{a_2b}\delta_{a_3b_1} + 3N^2\delta_{a_1b_1}\delta_{a_2b_3}\delta_{a_3b_4}\\
&+3N\delta_{a_1b_4}\delta_{a_2b_1}\delta_{a_3b_3} + N\delta_{a_1b_3}\delta_{a_3b_1}\delta_{a_2b_4} - 4\delta_{a_1b_1}\delta_{a_3b_3}\delta_{a_2b_4}]\\
&-\alpha^2(N^2-2)(\lambda_k)_{a_1b_1}(\lambda_k)_{a_4b_4}\delta_{a_2b_3} \qquad (16)
\end{aligned}$$

here we note that N is arbitrary integer,let $N = 3$, $\alpha = \frac{2}{3}$. That is, for the SU(3) case, with Eq.(10), one has

$$\delta_{b_1b_2}\delta_{a_2a_3} = \frac{1}{2}(\lambda_k)_{b_1a_3}(\lambda_k)_{a_2b_2} + \frac{1}{3}\delta_{b_1a_3}\delta_{a_2b_2} \qquad (17)$$

Consider $\delta_{b_1b_2}\delta_{a_2a_3}\delta_{b_3a_1}$:

$$\begin{aligned}
&\delta_{b_1b_2}\delta_{a_2a_3}\delta_{b_3a_1}\\
=\ &[\frac{1}{2}(\lambda_k)_{b_1a_3}(\lambda_k)_{a_2b_2} + \frac{1}{3}\delta_{b_1a_3}\delta_{a_2b_2}]\delta_{b_3a_1}\\
=\ &\frac{1}{4}(\lambda_k\lambda_l)_{b_1a_1}(\lambda_k)_{a_2b_2}(\lambda_l)_{b_3a_3} + \frac{1}{6}(\lambda_k)_{b_1a_1}(\lambda_k)_{a_2b_2}\delta_{b_3a_3}\\
&+\frac{1}{6}(\lambda_k)_{b_1a_1}(\lambda_k)_{b_3a_3}\delta_{a_2b_2} + \frac{1}{9}\delta_{b_1a_1}\delta_{a_2b_2}\delta_{b_3a_3} \qquad (18)
\end{aligned}$$

where one has made use of

$$\begin{aligned}
(\lambda_k)_{b_1a_3}\delta_{b_3a_1} &= (\lambda_k)_{b_1a_4}\delta_{a_4a_3}\delta_{b_3a_1}\\
&= (\lambda_k)_{b_1a_4}[\frac{1}{2}(\lambda_l)_{a_4a_1}(\lambda_l)_{b_3a_3} + \frac{1}{3}\delta_{a_4a_1}\delta_{b_3a_3}]\\
&= \frac{1}{2}(\lambda_k\lambda_l)_{b_1a_1}(\lambda_l)_{b_3a_3} + \frac{1}{3}(\lambda_k)_{b_1a_1}\delta_{b_3a_3} \qquad (19)
\end{aligned}$$

Substituting Eqs.(17,18) into Eq.(16), one has

$$
\begin{aligned}
&\beta_{kln}\beta_{nmo}\beta_{lmp}(\lambda_o)_{b_3a_3}(\lambda_p)_{b_1a_1}(\lambda_k)_{a_2b_2}\\
=\ &\frac{8}{27}[30\delta_{a_2a_3}\delta_{b_1a_1}\delta_{b_3b_2}-18\delta_{a_2a_1}\delta_{b_1a_3}\delta_{b_3b_2}-13\delta_{a_2b_2}\delta_{b_1a_1}\delta_{b_3a_3}\\
&-9\delta_{a_2a_3}\delta_{b_1b_2}\delta_{b_3a_1}+9\delta_{a_2b_2}\delta_{b_1a_3}\delta_{b_3a_2}+9\delta_{a_2a_1}\delta_{b_1b_2}\delta_{b_3a_3}]-\frac{28}{9}(\lambda_k)_{a_2b_2}(\lambda_k)_{b_3a_3}\delta_{b_1a_1}\\
=\ &\frac{8}{27}[30(\frac{1}{2}(\lambda_k)_{a_2b_2}(\lambda_k)_{b_3a_3}+\frac{1}{3}\delta_{a_2b_2}\delta_{b_3a_3}]\delta_{b_1a_1}-18[\frac{1}{4}(\lambda_k)_{b_1a_1}(\lambda_k\lambda_m)_{a_2b_2}(\lambda_m)_{b_3a_3}\\
+\ &\frac{1}{6}(\lambda_k)_{b_1a_1}(\lambda_k)_{a_2b_2}\delta_{a_3b_3}+\frac{1}{6}(\lambda_k)_{a_2b_2}(\lambda_k)_{b_3a_3}\delta_{b_1a_1}+\frac{1}{9}\delta_{b_1a_1}\delta_{a_2b_2}\delta_{b_3a_3}]\\
-\ &13\delta_{b_3a_3}\delta_{b_1a_1}\delta_{a_2b_2}-9[\frac{1}{4}(\lambda_k\lambda_m)_{b_1a_1}(\lambda_k)_{a_2b_2}(\lambda_m)_{b_3a_3}+\frac{1}{6}(\lambda_k)_{b_1a_1}(\lambda_k)_{a_2b_2}\delta_{b_3a_3}\\
+\ &\frac{1}{6}(\lambda_k)_{b_1a_1}(\lambda_k)_{a_2b_2}\delta_{b_3a_3}+\frac{1}{6}(\lambda_k)_{b_1a_1}(\lambda_k)_{b_3a_3}\delta_{a_2b_2}+\frac{1}{9}\delta_{b_1a_1}\delta_{a_2b_2}\delta_{a_3b_3}]\\
+\ &9[\frac{1}{2}(\lambda_k)_{b_1a_1}(\lambda_k)_{b_3a_3}+\frac{1}{3}\delta_{b_1a_1}\delta_{b_3a_3}]\delta_{a_2b_2}+9[\frac{1}{2}(\lambda_k)_{a_1b_2}(\lambda_k)_{b_1a_1}\\
+\ &\frac{1}{3}\delta_{b_1a_1}\delta_{a_2b_2}]\delta_{b_3a_3})-\frac{28}{9}(\lambda_k)_{a_2b_2}(\lambda_k)_{b_3a_3}\delta_{b_1a_1}\\
=\ &-\frac{4}{3}\beta^*_{klm}(\lambda_k)_{b_3a_3}(\lambda_l)_{b_1a_1}(\lambda_m)_{a_2b_2}-\frac{2}{3}\beta_{klm}(\lambda_k)_{b_3a_3}(\lambda_l)_{b_1a_1}(\lambda_m)_{a_2b_2}
\end{aligned}
\tag{20}
$$

Similarly, one can obtain

$$
\begin{aligned}
\beta_{klp}\beta_{kmn}\beta_{lmo}(\lambda_p)_{b_3a_3}(\lambda_n)_{b_1a_1}(\lambda_m)_{a_2b_2}=\ &-\tfrac{4}{3}\beta_{klm}(\lambda_k)_{b_3a_3}(\lambda_l)_{b_1a_1}(\lambda_m)_{a_2b_2}\\
&-\tfrac{2}{3}\beta^*_{klm}(\lambda_k)_{b_3a_3}(\lambda_l)_{b_1a_1}(\lambda_m)_{a_2b_2}
\end{aligned}
\tag{21}
$$

$$
\beta_{kln}\beta_{nmo}\beta_{mlp}(\lambda_o)_{b_3a_3}(\lambda_k)_{b_1a_1}(\lambda_p)_{a_2b_2}=4\beta_{klm}(\lambda_k)_{b_3a_3}(\lambda_l)_{b_1a_1}(\lambda_m)_{a_2b_2}
\tag{22}
$$

These relations have been checked numerically by a computer program.

3 DISCUSSION

These relations (Eqs.(20, 21 and 22)) may be useful in deriving many body interaction. For example, when we derive the qq$\bar{\text{q}}$–qq$\bar{\text{q}}$ interaction in the QCD[2], one needs Eq.(21).

The colour matrix element of the qq$\bar{\text{q}}$–qq$\bar{\text{q}}$ interaction can be written as:

$$
\begin{aligned}
&(\lambda^k\lambda^l)_{c_1'c_1}(\lambda^k)_{c_2'c_3'}(\lambda^l)_{c_3c_2} \\
&= \chi_{c_1'}\chi_{c_2'}\chi_{c_3}\left\{(\frac{2}{3}\delta_{kl}+\beta_{klp}\lambda_1^p)[\frac{1}{3}\lambda_2^k\lambda_3^l+\beta_{kmn}\beta_{kmo}\lambda_2^n\lambda_3^o\right. \\
&\left.+\frac{2}{9}\delta_{kl}+\frac{1}{3}\beta_{kln}\lambda_2^n+\frac{1}{3}\beta_{lko}\lambda_3^o]\right\}\chi_{c_1}\chi_{c_2}\chi_{c_3'} \\
&= \chi_{c_1'}\chi_{c_2'}\chi_{c_3}\left\{-\frac{2}{9}\vec{\lambda}_2\cdot\vec{\lambda}_3-\frac{4}{9}\vec{\lambda}_1\cdot\vec{\lambda}_2+\frac{4}{9}\vec{\lambda}_1\cdot\vec{\lambda}_3+\frac{32}{27}\right. \\
&\left.+\frac{1}{3}\beta_{klp}\lambda_1^p\lambda_2^k\lambda_3^l+\frac{1}{2}\beta_{klp}\beta_{kmn}\beta_{lmo}\lambda_1^p\lambda_2^n\lambda_3^o\right\} \\
&= \chi_{c_1'}\chi_{c_2'}\chi_{c_3'}(\frac{1}{9}(\lambda_2-\lambda_3^*)^2+\frac{16}{9}\lambda_1\cdot\lambda_3^*-\frac{2}{9}\lambda_1\cdot\lambda_2 \\
&-\frac{1}{3}\lambda_1\cdot\lambda_3^*\lambda_2\cdot\lambda_3^*-\frac{1}{3}\lambda_2\cdot\lambda_3^*\lambda_1\cdot\lambda_2)\chi_{c_1}\chi_{c_2}\chi_{c_3}
\end{aligned}
\tag{23}
$$

which has been made use of the generalized Fierz transformation and Eq.(21).

REFERENCES

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