Nonlinear feedback synchronization of hyperchaos in higher dimensional systems*

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Abstract Nonlinear feedback functional method is presented to realize synchronization of hyperchaos in higher dimensional systems. New nonlinear feedback functions and superpositions of linear and nonlinear feedback functions are also introduced to synchronize hyperchaos. The robustness of the method based on the flexibility of choices of feedback functions is discussed. By coupling well-known chaotic or chaotic-hyperchaotic systems in low-dimensional systems, such as Lorenz system, Van der Pol oscillator, Duffing oscillator and Rössler system, ten dimensional hyperchaotic systems are formed as the model systems. It can be found that there is not any noticeable difference in synchronization based on the numbers of positive Lyapunov exponents and of dimensions.

Keywords Hyperchaos, Synchronization, Nonlinear feedback functions, Ten-dimension.

1 Introduction

Recently, synchronization of hyperchaos in higher dimensional systems has become one of fundamental questions and much more important subject in chaos control theories and experiments due to a great potential possibility of applications.[1~14]. Examples of hyperchaos can be found[2~21] in systems such as arrays of Josephson junctions, coupled electric oscillators, artificial neural networks, arrays of chaotic systems, multi-mode lasers, and coupled map lattices. Synchronization of chaos and hyperchaos have been studied in the light of technical applications such as secure communications.[7,11,17,18]. It is very likely that control and synchronization of chaos and hyperchaos play important roles in the workings of biological and artificial neural networks. Although some progress has been made, an all-embracing unified method for synchronization of chaos and hyperchaos is missing. Fang[3~6], Kocarev *et al*[7], Pen *et al*[11] and Ali[[12~14] *et al* have recently studied synchronization of high-dimensional systems in which the transmitted signal is hyperchaotic[7,11], respectively. It is believed that communications via hyperchaotic signals are more secure and the procedure relates to the design of communication schemes by active-passive decomposition methods. Since hyperchaos exsits in nature, laboratory and the other social fields (such as economy, market), synchronization of hyperchaotic systems has therefore become a new area of active research and call for new methods. Since it is in general difficult to find a region in parameter space where hyperchaos exists, one method for synthesizing higher dimensional systems in a systematic way was proposed by Kocarev and Parliz.[7] Thus we only use standard low-dimensional systems with well-known dynamics to construct hyperchaotic systems. In the other words, hyperchaotic systems can be formed by coupling well known low-dimensional chaotic systems or chaotic-hyperchaotic systems.[7,22] Such a hyperchaotic system behaves[19,20] as a single chaotic or hyperchaotic system with new and rich dynamic behavior or it may behave as clusters of chaotic-hyperchaotic systems or it may even exhibit the behaviors of an assembly of individual chaotic-hyperchaotic systems. Quite often, a complex physical or biological system can be described in terms of the behaviors of

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simpler chaotic-hyperchaotic systems through a mechanism dubbed dynamic dissipation.[8] In this work, we study synchronization of hyperchaos in coupled chaotic-hyperchaotic systems using the method of nonlinear feedback functions which was applied to chaos as well as spatiotemporal chaos recently.[12~14]. So far, most of control and synchronization are based on the concept of linear feedback. The purpose of this paper is essentially to develop a nonlinear feedback functional method for synchronizing hyperchaos in very higher dimensional dynamical systems, such as the coupled Chua's circuits which has been extensively applied to many fields and the RLVPD with ten dimensional systems which was formed by well-known Lorenz system, Van der Pol oscillator, Duffing oscillator and Rössler system.

2 Variable feedback control

The method of variable feedback control has been used by a number of authors[2~29] to study control and synchronization of chaos and hyperchaos. Let us give a brief description here. We consider two n-dimensional autonomous dynamical systems described by the differential equations

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}) \tag{1}$$

$$\dot{\mathbf{Y}} = \mathbf{F}(\mathbf{Y}) + \mathbf{G}(\mathbf{X}, \mathbf{Y}) \tag{2}$$

here the state vectors $\mathbf{X}, \mathbf{Y} \in R^n$ are n-dimensional vectors with components $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$, respectively. The vector functions $\mathbf{F}$ and $\mathbf{G}$ have the components $F_1, F_2, \cdots, F_n$ and $G_1, G_2, \cdots, G_m$, $m \leq n$, respectively. The dynamical system of Eq.(1) is variably called the driver, master or sender system while the system of Eq.(2) is called the response, slave or receiver system. The function $\mathbf{F}(\mathbf{Y})$ is a replica of the function $\mathbf{F}(\mathbf{X})$ and $\mathbf{G}(\mathbf{X}, \mathbf{Y})$ is called the feedback function. Synchronization between the driver and response systems is said to be achieved if $\mathbf{X} = \mathbf{Y}$. An important property of the feedback function is that when synchronization has been realized, it vanishes, that is

$$\mathbf{G}(\mathbf{X}, \mathbf{Y}) = \mathbf{G}(\mathbf{X}, \mathbf{X}) = 0 \quad \text{when } \mathbf{X} = \mathbf{Y} \tag{3}$$

The state vector $\mathbf{X}(\mathbf{X} = \mathbf{X}(t))$ describes the 'goal' orbit for the response system to seek and follow. After synchronization has been achieved, the time evolution of the driver and the response systems are identical. The synchronization conditions $\mathbf{G}(\mathbf{X}, \mathbf{Y}) = \mathbf{G}(\mathbf{X}, \mathbf{X}) = 0$ when $\mathbf{X} = \mathbf{Y}$ can be satisfied by a very large (perhaps infinitely large) number of functional forms of $\mathbf{G}(\mathbf{X}, \mathbf{Y})$. Some simple forms of $\mathbf{G}(\mathbf{X}, \mathbf{Y})$ that have been used previously[23,11] are

$$G_i(\mathbf{X}, \mathbf{Y}) = K_i(Y_i - X_i) \tag{4}$$

$$G_i(\mathbf{X}, \mathbf{Y}) = B_i \sum_{j=1}^{n} K_j(Y_j - X_j); \quad i = 1, 2, \cdots, n \tag{5}$$

Eqs.(4) and (5) contain n and n^2 feedback parameters, as well as K_i, B_i, respectively. However, synchronization may be achieved by a fewer number of the parameters, in some cases synchronization is possible by adjusting only one parameter. We show below that linear as well as nonlinear forms of the feedback functions are effective in synchronizing hyperchaotic systems. The important point to note here is that the flexibility in the choice of $\mathbf{G}(\mathbf{X}, \mathbf{Y})$ provides robustness and efficiency to the method for dealing with chaotic and hyperchaotic systems of diverse nature. Obviously, a dynamical system can be synchronized by a variety of choices of the feedback function $\mathbf{G}(\mathbf{X}, \mathbf{Y})$. The main purpose of this work is to study some sample choices of the nonlinear feedback function to synchronize/control hyperchaos in two representative sets of coupled chaotic-hyperchaotic systems. According to our knowledge, most methods of variable feedback control of chaos have been restricted mainly to linear forms of the feedback function. In this work we have mainly used nonlinear forms of feedback functions. The linear ones and combinations with nonlinear ones are also applied to synchronize hyperchaos. In essence, the feedback function is a forcing term which can have varied nature and no wonder why various forms of the feedback function can synchronize a given hyperchaotic system. For representative samples of the nonlinear feedback functions (NFF), we have used the following kinds of form:

$$G_i(\mathbf{X},\mathbf{Y}) = K_i[1-e^{(Y_i-X_i)}] \ or \ [1-e^{K_i(Y_i-X_i)}] \tag{6}$$

$$G_i(\mathbf{X},\mathbf{Y}) = B_i \tanh[K_i(Y_i - X_i)] \tag{7}$$

$$G_i(\mathbf{X},\mathbf{Y}) = A_i \sin[K_i(Y_i - X_i)] \tag{8}$$

$$G_i(\mathbf{X},\mathbf{Y}) = K_i(Y_i - X_i) + B_i \tanh[K_i(Y_i - X_i)] + A_i \sin[K_i(Y_i - X_i)] \tag{9}$$

where A_i, B_i, and K_i, $i = 1, 2, \cdots, n$ are adjustable feedback parameters. Numerical results will show that all of the above forms of the feedback function and combinations of them can synchronize our model systems. Other forms of the feedback function are conceivable. The main idea of the algorithms of nonlinear feedback synchronization corresponding to the above NNF is that nonlinear feedback signal, $G_i(\mathbf{X},\mathbf{Y})$ corresponding to the NFF, is assigned to perturb the dynamical variables directly. The NFF is designed for the response system to be synchronized with the hyperchaos of the driver system. Therefore, the NFF tends to zero when the synchronization is realized. In the other words, the nonlinear signal forces the response system to be synchronized with one desired hyperchaos.

For the response system to synchronize, it is necessary that all of its Lyapunov exponents are negative.[17] Let $\delta\mathbf{Y}$ be a small change in $\mathbf{Y}$. Then in the linear approximation, we have the variational equation

$$\frac{d\delta\mathbf{Y}}{dt} = \nabla_{\mathbf{Y}}(\mathbf{F}(\mathbf{X},\mathbf{Y}) + \mathbf{G}(\mathbf{X},\mathbf{Y})) \bullet \delta\mathbf{Y} \tag{10}$$

here $\nabla_{\mathbf{Y}}$ represents the gradient with respect to $\mathbf{Y}$. For computing the Lyapunov exponents of the response system, Eqs.(1), (2) and (10) are solved simultaneously. Synchronization of chaos is achieved only if all of Lyapunov exponents of the response system are negative.[17] In fact, for measure of the efficiency of the method, the transient time of synchronization, τ is defined as the time required for $\Delta(t) = 0$ where

$$\Delta(t) = \sum_{i=1}^{n} |Y_i - X_i| \quad \text{or} \tag{11}$$

$$\Delta(t) = \sqrt{\sum_{i=1}^{n} (Y_i - X_i)^2} \tag{12}$$

τ depends on the method and its values of the parameters chosen. This is a qualifying measure for both of control and synchronization.

Control and synchronization are said to be achieved if the dynamical system describing the time evolution of the difference $\mathbf{e} = \mathbf{Y} - \mathbf{X}$, i.e, the error equation

$$\dot{\mathbf{e}} = \mathbf{F}(\mathbf{Y}) + \mathbf{G}(\mathbf{X},\mathbf{Y}) - \mathbf{F}(\mathbf{X})$$

$$= \mathbf{F}(\mathbf{X}+\mathbf{e}) + \mathbf{G}(\mathbf{X},\mathbf{X}+\mathbf{e}) - \mathbf{F}(\mathbf{X}) \tag{13}$$

has a stable fixed point at $\mathbf{e} = 0$. Another way of saying this is that we achieve synchronization if $||\mathbf{Y} - \mathbf{X}|| \to 0$ and $\mathbf{G}(\mathbf{X},\mathbf{Y}) \to 0$ as $t \to \infty$, where $\mathbf{X}$ is a desired (goal) state. The above synchronization conditions can be satisfied by a very large number of linear and nonlinear functional forms of $\mathbf{G}(\mathbf{X},\mathbf{Y})$. The feedback scheme of Eqs.(1-3) and Eq.(13) not only includes the approaches in current references and our suggestions but also has the flexibility of introducing all possible new feedback functions and thus giving robustness to the method of feedback control. For example, recently, Kocarev and Parlitz have used an active-passive decomposition scheme for synchronization.[7] This scheme decomposes the dynamical systems into

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X},\mathrm{s}) \tag{14}$$

$$\dot{\mathbf{Y}} = \mathbf{F}(\mathbf{Y},\mathrm{s}) \tag{15}$$

$$\mathrm{s} = \mathbf{h}(\mathbf{X}) \tag{16}$$

$$\dot{\mathrm{s}} = \mathbf{h}(\mathbf{X},\mathrm{s}) \tag{17}$$

Mathematically, this decomposition scheme is included in our approach if we choose

$\mathbf{G}(\mathbf{X}, \mathbf{Y}) = \mathbf{F}(\mathbf{Y}, s) - \mathbf{F}(\mathbf{Y})$. It has been shown the mathematical equivalence between their decomposition scheme and the approaches of Pecora and Carroll.[17,18] Therefore, the feedback scheme of Eqs.(1-3) is of universal formulation which includes control approaches as well as synchronization. The evidences are that a dynamical system can be synchronized by a variety of choices of the feedback function.

3 Models of hyperchaotic systems

Using the constructive method above, we have formed and studied two typical hyperchaotic systems in this work.

The first system is an unidirectionally-coupled Chua's circuits which is named the CCC and given by Ref.[1]

$$\dot{X}_1 = \alpha(X_2 - X_1 - f_x) \tag{18}$$

$$\dot{X}_2 = X_1 - X_2 + X_3 + K(X_5 - X_2) \tag{19}$$

$$\dot{X}_3 = -\beta X_2 \tag{20}$$

$$\dot{X}_4 = \alpha(X_5 - X_4 - f_u) \tag{21}$$

$$\dot{X}_5 = X_4 - X_5 + X_6 \tag{22}$$

$$\dot{X}_6 = -\beta X_5 \tag{23}$$

where

$$f_x = bX_1 + 0.5(a - b)(| X_1 + 1 | - | X_1 - 1 |) \tag{24}$$

$$f_u = bX_4 + 0.5(a - b)(| X_4 + 1 | - | X_4 - 1 |) \tag{25}$$

This is a six-dimensional hyperchaotic system. Here α, β, a and b are constants. The first Chua's circuit [Eqs.(18)-(20)] are coupled to the second one [Eqs.(21)-(23)] in such a way that the difference, $K(X_5 - X_2)$, between the signals X_5 and X_2, is introduced into the first circuit as a coupled term. The response system for the CCC model is defined as follows.

$$\dot{Y}_1 = \alpha(Y_2 - Y_1 - f_y) \tag{26}$$

$$\dot{Y}_2 = Y_1 - Y_2 + Y_3 + K(Y_5 - Y_2) \tag{27}$$

$$\dot{Y}_3 = -\beta Y_2 \tag{28}$$

$$\dot{Y}_4 = \alpha(Y_5 - Y_4 - f_v) \tag{29}$$

$$\dot{Y}_5 = Y_4 - Y_5 + Y_6 \tag{30}$$

$$\dot{Y}_6 = -\beta Y_5 \tag{31}$$

where

$$f_y = bY_1 + 0.5(a - b)(| Y_1 + 1 | - | Y_1 - 1 |) \tag{32}$$

$$f_v = bY_4 + 0.5(a - b)(| Y_4 + 1 | - | Y_4 - 1 |) \tag{33}$$

For the 6-dimensional CCC system with fixed parameters ($\alpha = 10$, $\beta = 14.87$, $a = -1.27$, $b = -0.68$, $k = 0.02$), we have obtained Lyapunov exponential spectra (0.4335, 0.4125, 0.0009, 0.00000, -3.7522, -3.8741), in which there are three positive ones. It was also shown in Ref.[1] the robustness of the hyperchaos in the CCC way in which it may be possible to obtain hyperchaotic attractors with N positive Lyapunov exponents in a chain of N unidirectionally-coupled Chua's circuits. But they have not shown synchronization of hyperchaos yet. Could we realize it? That is our subject in this work.

The second system with ten dimensions called the RLVPD system is formed by coupling well-known Rössler hyperchaotic system, Lorenz system and Van der Pol-Duffing oscillator.[19] The differential equations for the RLVPD system are given by

$$\dot{X}_1 = -\nu(X_1^3 - \alpha X_1 - X_2) \tag{34}$$

$$\dot{X}_2 = X_1 - X_2 - X_3 + C_1 X_7 \tag{35}$$

$$\dot{X}_3 = \beta X_2 \tag{36}$$

$$\dot{X}_4 = -\sigma_1(X_4 - X_5) \tag{37}$$

$$\dot{X}_5 = \rho X_4 - X_5 - X_4X_6 + C_2X_7 + C_3X_8 \quad (38)$$

$$\dot{X}_6 = X_4X_5 - bX_6 \quad (39)$$

$$\dot{X}_7 = -X_8 - X_9 + C_4X_4 + C_5X_1 \quad (40)$$

$$\dot{X}_8 = X_7 + 0.25X_8 + X_{10} \quad 41)$$

$$\dot{X}_9 = 3 + X_7X_9 \quad (42)$$

$$\dot{X}_{10} = -0.5X_9 + 0.05X_{10} \quad (43)$$

The response system for the RLVPD model is defined in the same way as in the CCC case. X_1, X_4, X_7 and X_8 are used for couplings of the RLVPD system. The couplings are chosen such that for a range of random initial conditions the coupled systems remain bounded, possess two or more positive Lyapunov exponents, and exhibit global chaotic behavior. The couplings shown above are found satisfactory for the purpose of our study. The parameter values used for the RLVPD system are given by $\nu = 300$, $\alpha = 0.35$, $\beta = 45$, $\rho = 28$, b=8/3, $\sigma_1 = 10$. The initial conditions are as follows: using random numbers for X_1, X_2 and X_3, and the others are zero for the driver system but using random numbers for Y_3, $Y_5 = 1$ and the others are zero in the response system. The RLVPD has two or more positive Lyapunov exponents for a large set of values of the coupling parameters $C_i, i = 1, 2, 3, 4, 5$. For examples, for $C_1 = 1, C_2 = 1, C_3 = 37$ and $C_4 = C_5 = 0$, the Lyapunov exponential spectra for the RLVPD are three positive ones: (0.198, 0.333, 0.008), a 0.000, and six negative (-1.599 ,-2.999, -32.874, -701.014, ...). For $C_1 = 1, C_2 = 1, C_3 = 20$ and $C_4 = C_5 = 0$, the RLVPD has two positive Lyapunov exponents (0.262, 0.031) and a zero and seven negative ones. Our numerical results show that many sets of values of the C_i can be used for the study of hyperchaos in the RLVPD system. Note that the RLVPD is a ten-dimensional hyperchaotic system which has been never investigated yet.

4 Synchronization of hyperchaos

Based on the above model hyperchaotic systems, we are in a position to focus on the main issue of synchronization of hyperchaos. The variable feedback control method is robust in the sense that a given dynamical system can be synchronized with its replica by a variety of choices of forms of the feedback function $\mathbf{G}(\mathbf{X}, \mathbf{Y})$ and by a variety of choices of parameter values in each functional form. This implies that temporal synchronization of hyperchaos becomes simpler when the method of variable feedback functions is employed.

Now, let us see the results of hyperchaotic synchronization in the CCC and RLVPD systems by various NFF, respectively. For illustrating synchronization of hyperchaos in the model CCC and RLVPD by the NFF, we have calculated all the Lyapunov exponents of the synchronized systems and the transition time. All of Lyapunov exponents of the synchronized systems are found to be negative. Such as, for the replica of the CCC system (18-23) synchronized by the NFF $G_2 = -10\tan h(5(Y_1 - X_1)$ and $G_5 = -10\tan h(5(Y_5 - X_5))$, the Lyapunov exponents are (0.0000, -0.0004, -0.0007,-0.0018, -0.2918, -50.7053).

For the synchronized replica of the RLVPD (Eqs.(34-43)) under the above two groups of connection coefficient (1) $C_1 = 1, C_2 = 1, C_3 = 37, C_4 = C_5 = 0$ and (2) $C_1 = 1, C_2 = 1, C_3 = 20, C_4 = C_5 = 0$, and $K_1 = 12$, $K_2 = 13$, $K_3 = 14$, $K_4 = 15$, $K_5 = 16$, $K_6 = 17$, $K_7 = 500$, $K_8 = 880$, $K_9 = 2000$, $K_{10} = 2001$, after nonlinear feedback by the NFF of No.3 in Table 1, the Lyapunov exponents are (-13.85,-13.94,-17.69, -17.97, -26.04, -499. 99, -877.91, -1769.40, -2020.15, -2794.07) and (-13.49,-13.87, -17.86, -18.23, -25.58, -497.45, -867.83, -915.88, -2000.95,-2024.65), respectively. These evidences show that hyperchaotic synchronization is realized successfully and all of hyperchaos for our models is synchronized very well by the NFF.

For measuring efficiency of synchronizing hyperchaos, we have also computed the average transition time , τ, over 100 initial conditions for each choice of the feedback function. The results are given in Tabs.1-2 for above two hyperchaotic models. It is seen from Table 1 that the No.8 of NFF can realize synchronization with faster transition time than the linear feedback one. Another interesting trend is that the average transition time τ in the case of combinations of the NFF is always smaller than the average τ of the constituents of the combinations. For example, No.8 of the NFF in Tab.2 is of fastest transition time than the other NFF above since it has more superposition of the NFF and linear one. The more superposition, the faster transition time. The nonlinear feedback functions and their superposition (including with linear one) can work much more efficiently for synchronization of hyperchaos. The above results may provide a way to expedite synchronization in real experimental situation.

Table 1 Transition time τ versus some feedback functions in the CCC system for the above parameters in the text

No	Feedback function $\mathbf{G(X,Y)}$	τ CCC
1	$G_i = -100(Y_i - X_i), i = 2,5$	12
2	$G_i = -15\tanh(5(Y_i - X_i)), i = 2,4;$	9
3	$G_2 = -100(Y_2 - X_2), G_4 = -100\sin(0.25(Y_4 - X_4))$	11
4	$G_2 = -50\tanh(1.5(Y_2 - X_2)), G_4 = -100\sin(0.25(Y_4 - X_4))$	11
5	$G_2 = -100\sin(0.5(Y_2 - X_2)), G_5 = -100(Y_5 - X_5)$	15
6	$G_2 = -5\sin(15(Y_2 - X_2)), G_5 = -5\tanh(5(Y_4 - X_4))$	11
7	$G_2 = -25\tanh(1.5(Y_2 - X_2) - 0.05(3(Y_2 - X_2)^2 - 1)), G_5 = -5(Y_5 - X_5)$	11
8	$G_2 = -10\tanh(5(Y_1 - X_1)), G_5 = -10\tanh(5(Y_5 - X_5))$	1

Table 2 Comparison of average transition time τ over 100 initial conditions for various forms of the feedback functions in the RLVPD systems. The feedback parameters values are as follows: $A = 100$, $K_1 = 12$, $K_2 = 13$, $K_3 = 14$, $K_4 = 15$, $K_5 = 16$, $K_6 = 17$, $K_7 = 500$, $K_8 = 880$, $K_9 = 2000$, $K_{10} = 2001$.

No	Feedback function $\mathbf{G(X,Y)}$	τ RLVPD
1	$1 - e^{K_i(Y_i - X_i)}$	170.287
	$K_i(1 - e^{(Y_i - X_i)})$	2.664
2	$-\tanh K_i(Y_i - X_i)$	7.038
3	$-A\sin K_i(Y_i - X_i)$	165.50
4	$-\tanh K_i(Y_i - X_i) - A\sin K_i(Y_i - X_i)$	2.14
5	$K_i(Y_i - X_i) - A\sin K_i(Y_i - X_i)$	2.008
6	$K_i(Y_i - X_i) - \tanh K_i(Y_i - X_i)$	1.405
7	$K_i(Y_i - X_i) - \tanh K_i(Y_i - X_i) - A\sin K_i(Y_i - X_i)$	1.175
8	$K_i(Y_i - X_i) + K_i(1 - e^{(Y_i - X_i)})$ $-\tanh K_i(Y_i - X_i) - A\sin K(Y_i - X_i)$	0.812

5 Discussion and conclusion

In this paper we have presented the nonlinear feedback functional method to realize synchronization of hyperchaotic systems in higher dimensional dynamical systems. For the model hyperchaotic systems, we have introduced the six-dimensional and ten-dimensional system by coupling (1) the coupled Chua's curcuits (CCC system) and (2) the coupled Rössler hyperchaotic attractor, Lorenz attractor and Van der Pol-Duffing oscillator (RLVPD system). The couplings gave us hyperchaotic systems possessing two and three positive Lyapunov exponents. The number of positive Lyapunov exponents of a coupled system is not equal to the sum of the number of positive Lyapunov exponents of its components. We have studied synchronization of cases involving two and three positive Lyapunov exponents. Our main focus has been to illustrate the flexibility and robustness in the method of nonlinear feedback control by considering several examples of nonlinear and linear feedback functions and their superpositions. Main numerical results for the cases that we have studied are as follows. (1)For feedback both linear and nonlinear feedback functions can synchronize hyperchaotic systems. All of the NFF can reach the hyperchaotic synchronization but some of the NFF which can be found is luckier and better than linear one because it has faster transition time and only one or two NFF applied to one or two equations of response system are enough for synchronization. (2)The feedback functions obtained by superposition of known linear and nonlinear feedback functions can also synchronize the same hyperchaotic system as well as they are more efficent than their each component does. One can explore larger classes of feedback functions to synchronize hyperchaos. We anticipate that a realization of the existence of such a flexibility will help design appropriate feedback controls in real experimental situations. From the point of view of synchronization by variable feedback control, we do not find any difference depending on the numbers of positive Lyapunov exponents and of dimensions. We have considered here the flexibility in synchronization but the same flexibility also exists for both control and synchronization of spatiotemporal chaos.[12~14] This method therefore has the potential of practical applications. For example, in secure communication suggested in Refs.[7,11] hyperchaotic signals are transmitted to mask a message and a synchronized receiver system is set to recover the message. Thus it may improve security and obtain a more efficient encoding of information.

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