The measuring significance of vierbein*

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Abstract The problem of what is observable in general relativity is investigated. It is proposed that what is observable in curved spacetime is Lorentz quantity instead of Riemann quantity. With the help of Landau's observable space interval, the observational frames for individual observers are also established.
Keywords Lorentz quantity, Observation frame, General relativity, Space-time

1 Introduction

The problem of what precisely is observable is far from trivial in general relativity. Actually, one may find in the literature substantially non-overlapping viewpoints on this issue[1]. Recently, C. Rovelli considered in the metric representation a model in which the matter represents the physical reference system[2]. Here we discuss the problem of observables in the vierbein formulation.

It is well-known that in the vierbein formalism of general relativity, there are two kinds of quantities: Riemannian quantity such as the curvature $R_{\mu\nu\alpha\beta}$ and the Lorentz quantity such as R_{abmn} related to $R_{\mu\nu\alpha\beta}$, i.e.

$$R_{abmn} = e^{\mu}_{a}e^{\nu}_{b}e^{\alpha}_{m}e^{\beta}_{n}R_{\mu\nu\alpha\beta} \qquad (1)$$

Our conventions are as follows: $g_{\mu\nu} = \eta_{ab}e^{\alpha}_{\mu}e^{b}_{\nu}$, η_{ab} =diag(1, −1, −1, −1). Then, a problem arises: which kind of quantities is observable? For instance, when there exists a Maxwell field, which one of $F_{\mu\nu}$ and F_{ab} is measured in one's laboratory? This problem is very important because physics, by its nature, is to study the observables and their relationship.

2 The observation frame

Since the measuring of space and time is the foundation of all other measuring, so one should study first the observable time and spatial interval of two world events x and $x + \mathrm{d}x$ whose interval is $\mathrm{d}x^{\mu}$. A. Einstein once pointed out that in a local system, for two neighbouring point events

$$\mathrm{d}x^2 = g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = \Delta T^2 - \Delta L^2 \qquad (2)$$

where ΔL can be measured directly by a measuring rod and ΔT by a clock at rest relative to the system: these are the naturally measured lengths and times[3]. Though there is still an ambiguity in this statement which is that ΔT and ΔL are different in different Lorentz gauges, one point is clear: what is measured is ΔT and ΔL instead of $\mathrm{d}x^{\mu}$. About time, S. Hawking once said that in the theory of relativity, there is no unique absolute time, but instead each individual has his own personal measure of time that depends on where he is and how he is moving[4]. This arguement is in good agreement with that of Einstein. Moreover, it pointed out that an observer is characterized by his position and velocity, i.e. an observer can be denoted by $\mathcal{O}(x, u)$. About physical measurement, R. M. Wald had once pointed out that an observer, equiped with measuring apparatus, can be characterized by an orthonormal basis, which forms an locally inertial frame, e^{a}_{μ}, of which the first denotes the alignment of his time rod and the other three serve as references for how he aligns the measuring apparatus[5]. So according to this arguement, the observed quantities are the projections onto the basis e^{a}_{μ} of the quantities in general spacetime. Therefore, the key problem is how to determine the basis.

Let us study first the case for a static observer $\mathcal{O}$ whose velocity $u^{\mu} = (g_{00}^{-1/2}, 0, 0, 0)$. The measured time and space interval for a static observer have been given by Landau[6]

$$\Delta T = \frac{g_{0\mu}}{\sqrt{g_{00}}}\mathrm{d}x^{\mu} \qquad (3)$$

$$\Delta L^2 = (\frac{g_{0i}g_{0j}}{g_{00}} - g_{ij})\mathrm{d}x^{i}\mathrm{d}x^{j} \qquad (4)$$

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We next show that this is equivalent to say that the time rod of $\mathcal{O}$ is fixed by

$$e^0_\mu = u_\mu \tag{5}$$

Since

$$u_\mu = g_{\mu\nu}u^\nu = \frac{g_{\mu 0}}{\sqrt{g_{00}}} \tag{6}$$

So if Eq.(5) holds, from $(a' = 1, 2, 3)$

$$g_{00} = (e^0_0)^2 - (e^{a'}_0)^2 \tag{7}$$

one has

$$e^{a'}_0 = 0 \tag{8}$$

so from Eq.(1), one has Eqs.(3,4). Eqs.(5,7) determine the time rod of $\mathcal{O}$, the directions of the spatial rods remain arbitrary, i.e. he may rotate arbitrarily his apparatus and this does not affect his observed results.

For a moving observer $\mathcal{O}'$, how to fix the basis? As a principle, Eq.(5) is generalized to the case for arbitrary u^μ. The reason for this generalization is that it makes the observed time and length independent coordinates x^μ. Suppose his velocity is $u'^\mu \neq u^\mu$, the basis is denoted as $\bar{e}^a_\mu$, then according to special relativity, $\bar{e}^a_\mu$ should be related to e^a_μ by a local Lorentz transformation, which is determined by the relative Lorentz velocity $u'^a = u'^\mu e^a_\mu$

$$\bar{e}^a_\mu(x) = \Lambda^a{}_b(u'^a)e^b_\mu(x) \tag{9}$$

where $\Lambda^a{}_c\Lambda^b{}_d\eta_{ab} = \eta_{cd}$. Thus Λ is determined up to a spatial rotation,

$$\Lambda^0{}_b = u'_b \tag{10}$$

With this definition of observable space and time interval, we can solve the problems of twin paradox and red shift. Let us discuss the latter. Consider two static observers A and B, two light waves start at $x^\mu_{\rm A}$ and $x^\mu_{\rm A} + {\rm d}x^\mu_{\rm A}$ to travel from A to B. The world lines for each wave, C_1, C_2 are null, so

$$\mathrm{d}t = \frac{-g_{0i}\mathrm{d}x^i + \sqrt{(g_{0i}g_{0j} - g_{00}g_{ij})\mathrm{d}x^i\mathrm{d}x^j}}{g_{00}} \tag{11}$$

In general, Eq.(11) is a differential equation, in some special cases, for example, the static metric and the Robertson-Walker metric, it can be readily integrated. After Eq.(11) is solved, one can obtain the world points $x^\mu_{\rm B}$ and $x^\mu_{\rm B} + {\rm d}x^\mu_{\rm B}$ at which the two waves arrive at B. Then the red shift is

$$\frac{\nu_{\rm B}}{\nu_{\rm A}} = \sqrt{\frac{g_{00}(x_{\rm B})}{g_{00}(x_{\rm A})}}\frac{g_{0\mu}(x_{\rm A})\mathrm{d}x^\mu_{\rm A}}{g_{0\mu}(x_{\rm B})\mathrm{d}x^\mu_{\rm B}} = \sqrt{\frac{g_{00}(x_{\rm A})}{g_{00}(x_{\rm B})}}\frac{\mathrm{d}x^0_{\rm A}}{\mathrm{d}x^0_{\rm B}} \tag{12}$$

For static metric such as Schwarzchild metric, it can be easily shown that ${\rm d}x^0_{\rm A} = {\rm d}x^0_{\rm B}$, so one has

$$\frac{\nu_{\rm B}}{\nu_{\rm A}} = \sqrt{\frac{g_{00}(x_{\rm A})}{g_{00}(x_{\rm B})}} \tag{13}$$

For the time-dependent R-W metric, one has

$$\frac{\mathrm{d}x^0_{\rm A}}{\mathrm{d}x^0_{\rm B}} = \frac{R(x^0_{\rm A})}{R(x^0_{\rm B})} \tag{14}$$

so the red shift is given by

$$\frac{\nu_{\rm B}}{\nu_{\rm A}} = \frac{R(x^0_{\rm A})}{R(x^0_{\rm B})} \tag{15}$$

Though the arguement that the quantity $F_{\mu\nu}$ is not observable may sound strange, it can be illustrated by the following example. Suppose that there is a uniform electric field $\vec{E} = (0, 0, E)$ in the Minkowski spacetime. An observer, S, rotated around the z-axis with angular velocity ω, the distance between S and the z-axis is r, $r\omega \le c$. Denote the coordinates of the static frame as $X^\mu = (cT, R, \Theta, Z)$, of the rotating frame as $x^\mu = (ct, r, \theta, z)$. Then,

$$\begin{aligned} R = r, \quad Z = z, \\ T = t, \quad \theta = \Theta + \omega t \end{aligned} \tag{16}$$

hence

$$\begin{aligned} \mathrm{d}s^2 &= c^2\mathrm{d}T^2 - \mathrm{d}R^2 - R^2\mathrm{d}\Theta^2 - \mathrm{d}Z^2 \\ &= G_{\mu\nu}\mathrm{d}X^\mu\mathrm{d}X^\nu \\ &= (1 - \omega^2r^2/c^2)c^2\mathrm{d}t^2 - \mathrm{d}r^2 \\ &-r^2\mathrm{d}\theta^2 - \mathrm{d}z^2 + \tfrac{2\omega r^2}{c}\mathrm{d}\theta\mathrm{d}(ct) = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \end{aligned} \tag{17}$$

so to S, the spacetime is no longer Minkowski. According to the discussion above, the observation frame of S is given by

$$e^a_\mu(x) = \begin{pmatrix} \sqrt{1 - \frac{\omega^2r^2}{c^2}} & 0 & \frac{\omega r^2}{\sqrt{c^2-\omega^2r^2}} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{rc}{\sqrt{c^2-\omega^2r^2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{18}$$

In the static frame, the electromagnetic tensor is

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -E/c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ E/c & 0 & 0 & 0 \end{pmatrix} \tag{19}$$

Hence, the electromagnetic tensor in the rotating frame is given by

$$f^{\mu\nu} = \frac{\partial x^\mu}{\partial X^\alpha}\frac{\partial x^\nu}{\partial X^\beta}F^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & -E/c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega E/c^2 \\ E/c & 0 & \omega E/c^2 & 0 \end{pmatrix} \tag{20}$$

i.e.

$$f^{ab} = \begin{pmatrix} 0 & 0 & 0 & -\gamma E/c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma\omega rE/c^2 \\ \gamma E/c & 0 & \gamma\omega rE/c^2 & 0 \end{pmatrix} \tag{21}$$

where $\gamma = 1/\sqrt{1-\frac{r^2\omega^2}{c^2}}$. That is the electromagnetic field S observed is

$$\begin{aligned} \vec{E} &= (0,0,\gamma E) \\ \vec{B} &= (\gamma\omega rE/c^2,0,0) \end{aligned} \tag{22}$$

According to special relativity, the relationship of electromagnetic fields in two inertial frames is given by

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel}, \quad \vec{B}'_{\parallel} = \vec{B}_{\parallel} \tag{23}$$

$$\begin{aligned} \vec{E}'_{\perp} &= \gamma(\vec{E}+\vec{\nu}\times\vec{B})_{\perp}, \\ \vec{B}'_{\perp} &= \gamma(\vec{B}-\frac{\vec{\nu}}{c^2}\times\vec{E})_{\perp} \end{aligned} \tag{24}$$

when $\nu \to c$, $\vec{E}_{\perp} \to \infty$, $\vec{B}_{\perp} \to \infty$, hence it is obvious that $f^{\mu\nu}$ is not the observed quantity, whereas f^{ab} is.

3 Conclusion

Finally, we make some remarks. First, according to Eq.(1), the observed speed of light is always constant and isotropic. Second, it is necessary to introduce vierbein formalism not only because of mathematical reasons but also because of physical measurement. Third, the general covariant conservative energy-momentum and angular-momentum obtained in Refs.[7~ 9] are actually observable.

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